MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. V.

From the Editor

MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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CONTENTS.

Mathematical Papers, &c.

Solved Questions.

Unsolved Questions.

1448. Proposed by W. K. Clifford, Trinity College, Cambridge.)—The sides of a triangle repel with a force varying inversely as the cube of the distance; find the position in which a particle will rest.

Also, supposing the faces of a tetrahedron to repel according to the same law, find where a particle will rest.

1524. (Proposed by the Rev. T. P. Kirkman, M.A., F.R.S.)—Let N be any odd number, and e_1, e_2, e_3...different positive numbers; also let

$$N-N' = 2m_1e_1 + 2m_2e_2 + 2m_3e_3 + \&c. \quad \ldots\ldots\ldots\ldots\ldots(A)$$

be in turn every partition of that form of every even number $\overline{\overline{>}}$ 0 and < N, ($N-N' = 2 \cdot 0 \cdot e_1$ being the case of $N' = N$); then

$$\frac{N(N-2)(N-4)\ldots\ldots 3 \cdot 1}{(N-1)(N-3)\ldots\ldots 4 \cdot 2} = \Sigma \frac{1}{(2e_1)^{m_1} \cdot \Pi m_1 \cdot (2e_2)^{m_2} \cdot \Pi m_2 \ldots}$$

(Πm being $1 \cdot 2 \ldots m$, and $\Pi 0 = 1$), where every partition (A) gives a term of the sum Σ, and all partitions are different which have not the same e_1, e_2, e_3...(thus $9-1 = 2 \cdot 4 \cdot 1 = 2 \cdot 1 \cdot 4 = 2 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2$ are all different partitions.)

1591. (Proposed by Professor Hirst.)—Find the polar equation of a curve whose radii vectores are each divided into segments having a *constant* ratio when, upon the same, the respective centres of curvature are projected orthogonally.

1787. (Proposed by Professor Cremona.)—On donne une conique K et un point p. Une transversale menée arbitrairement par p rencontre K en deux points m, m'; et soit x un point de la transversale tel que le rapport anharmonique $(pxmm')$ soit un nombre λ donné. Trouver le lieu du point x. Si λ est l'une des racines cubiques imaginaires de -1, on a une certaine conique C (p). De quelle manière change C (p), si l'on fait varier p?

Recherche analogue par rapport à une surface du second ordre.

No.

1796. (Proposed by J. CASEY, B.A.)—Prove that (1) if any two circles be inverted from an arbitrary point, the ratio of the square of their common tangent to the rectangle of their diameters, is the same for the converse as for the original circles. (2) Employ the preceding theorem in proving Terquem's theorem on the nine-point circle, giving a proof that will apply equally whether the triangle be formed by right lines or by arcs of circles. (3) Also prove Terquem's Theorem for spherical triangles formed by arcs of great or small circles.

1797. (Proposed by W. S. BURNSIDE, B.A.)—Determine a property of the surface which renders the double integral

$$\iint \left\{ \mathrm{N}\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}_1}\right)-\frac{\mathrm{P}^2}{\mathrm{RR}_1}\right\} dx\,dy \text{ a maximum;}$$

where N is the normal, P the perpendicular from the origin on the tangent plane, and R, R_1 the principal radii of curvature.

1819. (Proposed by C. TAYLOR, M.A.)—From two fixed points on a given conic pairs of tangents are drawn to a confocal conic, and with the fixed points as foci a conic is described passing through any one of the four points of intersection. Show that its tangent or normal at that point passes through a fixed point.

1831. (Proposed by PAUL SERRET.)—Une ellipse et l'un de ses cercles directeurs étant tracés, il existe une infinité de triangles simultanément inscrits au cercle et circonscrits à l'ellipse ; le point de rencontre des hauteurs est le même pour tous ces triangles.

1832. (Proposed by Professor SYLVESTER.)—Find the chance that if three points be taken at random inside a circle, any two of them shall be nearer to one another than the remaining one to the centre.

1833. (Proposed by Professor SYLVESTER.)—(1.) If r represent the mutual distance of any two points in a triangle from each other; r_1, r_2, r_3 the distances of points in the three sides respectively from the opposite angles; and if

$$\psi r = \frac{8}{r^4}\left(\int_0^r dr\right)^3 r\phi r; \text{ prove that the mean value of } \phi r,$$

when within the limits of the figure it is expressible by a series of ascending powers of r, is equal to the sum of the mean values of ψr_1, ψr_2, ψr_3.

(2.) If r represent the mutual distance of any two points in a tetrahedron from each other; r_1, r_2, r_3, r_4 the distances of points in the faces respectively from the opposite angles; ρ_1, ρ_2, ρ_3 the mutual distances of points in the pairs of opposite

$$\text{edges; and if } \psi r = \frac{36}{r^6}\left(\int_0^r dr\right)^4 r^2\phi r, \phi \text{ being subject to the}$$

same condition as above; prove that the mean value of ϕr is equal to the sum of the mean values of ψr_1, ψr_2, ψr_3, ψr_4 together with twice the sum of the mean values of $\psi\rho_1$, $\psi\rho_2$, ψr_3.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

1791. Proposed by Professor CAYLEY.)—Given a quartic curve $U = 0$, to find three cubic curves $P = 0$, $Q = 0$, $R = 0$, each meeting the quartic in the same six points 1, 2, 3, 4, 5, 6, and such that $P=0$, $R=0$ may besides meet the quartic in the same three points a, b, c, and that $Q=0$, $R=0$ may besides meet the quartic in the same three points α, β, γ.

Solution by PROFESSOR CREMONA.

Considérons le problème comme résolu. Les courbes U, R ont en commun les douze points $123456abc\alpha\beta\gamma$, dont les neufs premiers sont situés dans une cubique P; donc les trois derniers seront en ligne droite. Analoguement, les six premiers points et les trois derniers étant dans une cubique Q, il ensuit que abc sont en ligne droite. Alors, les deux cubiques P, R ayant en commun trois points abc en ligne droite, les six autres intersections 123456 appartiendront à une même conique. Soient mn, o, ω les nouvelles intersections de U avec cette conique et les droites abc, $\alpha\beta\gamma$. La courbe U et l'autre ligne du quatrième ordre formée par la conique et les deux droites susdites ont douze points communs placés sur la cubique R; donc les quatre autres points communs $mno\omega$ seront en ligne droite.

Cela posé, on resoudra le problème de la manière suivante. Coupons la courbe donnée U par une conique, et soient $123456mn$ les intersections résultantes. La droite mn rencontrera de nouveau U en deux points $o\omega$. Par o et par ω conduisons deux droites qui coupent U respectivement en abc, $\alpha\beta\gamma$. Alors, la courbe U et la ligne du quatrième ordre formée par la conique 123456 et par les droites $oabc$, $\omega\alpha\beta\gamma$ auront en commun quatre points $mno\omega$ en ligne droite; donc on pourra tracer une cubique R par les douze autres intersections $123456abc\alpha\beta\gamma$. La cubique R et la ligne du troisième ordre formée par la conique susdite et par la droite abc ont en commun les neuf points $123456abc$ de U; donc on peut faire passer par ces points une autre cubique P. De même, on peut tracer une autre cubique Q par les neufs points $123456\alpha\beta\gamma$.

1792. (Proposed by R. WARREN, B.A.)—Find the condition in order that the normals to the conic $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}}$, drawn at the points of contact of the sides of the triangle of reference, may meet in a point.

Solution by the EDITOR.

The normal perpendicular to the side γ passes through the point $(\gamma, l\alpha - m\beta)$, hence, writing (λ, μ, ν) for $(\cos A, \cos B, \cos C)$, the trilinear equation of this normal is readily found to be $l\alpha - m\beta + (l\mu - m\lambda)\gamma = 0$.

The equations of the other two normals are of course similar to this; and the condition of the concurrence of the three is

$$\begin{vmatrix} l, & -m, & l\mu - m\lambda \\ m\nu - n\mu, & +m, & -n \\ -l, & n\lambda - l\nu, & +n \end{vmatrix} = 0,$$

which, when expanded, may be expressed in the form

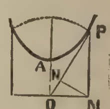
$$a^2mn(m\nu - n\mu) + b^2nl(n\lambda - l\nu) + c^2lm(l\mu - m\lambda) = 0.$$

The locus of the centres of the conics which satisfy this condition of concurrence is a cubic possessing many interesting properties: for which see Quest. 1545 (*Reprint*, Vol. II., p. 57), also a paper in the *Messenger of Mathematics*, Vol. III., p. 15.

1813. (Proposed by the late H. J. PURKISS, B.A.)—A luminous surface of uniform intrinsic brightness B is of the form generated by the revolution of a catenary about its axis; prove that the illumination of a small plane area δ situated at the intersection of the axis and directrix of the catenary, and having its plane perpendicular to the axis, is

$$\frac{4\pi B\delta}{(e^{\alpha} + e^{-\alpha})^2}, \ \alpha \text{ being such that } e^{2\alpha} = \frac{\alpha+1}{\alpha-1}.$$

Solution by the PROPOSER; H. M. TAYLOR, B.A.; J. DALE; *the* REV. J. L. KITCHIN, M.A.; *and others.*



Let OA be the axis, OM the directrix of the generating catenary. Let OP be drawn to touch the curve at P. Draw the ordinate PM, and MN perpendicular to OP. Let OM $= x$, PM $= y$, OA $=$ MN $= c$, $\frac{x}{c} = \alpha$. Then the illumination at O is the same as would be produced by the portion of a spherical surface (of the same intrinsic brightness), having O for its centre and OP for its radius, bounded by the conical surface formed by the revolution of OP about OA. Now the projection of this portion of surface on the plane to be illuminated is a circle of radius OM. Hence the expression for the illumination will be

$$\frac{\pi \cdot OM^2 \cdot B \cdot \delta}{OP^2} = \frac{\pi B\delta \cdot MN^2}{MP^2} = \pi B\delta\left(\frac{c}{y}\right)^2 = \frac{4\pi P\delta}{(e^{\alpha} + e^{-\alpha})^2}.$$

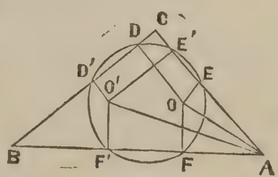
To determine α, we have clearly at P

$$\frac{y}{x}=\frac{dy}{dx}=\frac{e^{\alpha}-e^{-\alpha}}{2}; \therefore \frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}=\alpha; \therefore e^{2\alpha}=\frac{\alpha+1}{\alpha-1}.$$

This equation has only one real positive root, which is greater than unity. For $e^{2\alpha}$ is always positive, whereas $\frac{\alpha+1}{\alpha-1}\left(=1+\frac{2}{\alpha-1}\right)$ is only positive when $\alpha>1$, and we see moreover that it diminishes as α increases. These results are of course obvious from the geometry of the figure.

1815. (Proposed by J. GRIFFITHS, M.A.)—Show that the feet of the perpendiculars drawn from the two points (l, m, n), (l^{-1}, m^{-1}, n^{-1}), upon the sides of their triangle of reference, all lie on the same circle; and find its equation.

Solution by H. M. TAYLOR, B.A.



Let any circle cut the sides BC, CA, AB of a triangle in D, D′; E, E′; F, F′; from E, E′ and F, F′ draw perpendiculars to CA, AB meeting in O, O′; and join AO, AO′. Let $\angle$OAE $=\theta$, OAF $=\phi$, O′AE′ $=\theta'$, O′AF′ $=\phi'$; then $\theta+\phi=\text{A}=\theta'+\phi'$, or $\theta'-\theta=\phi-\phi'$, whence $\cos\theta'\cos\theta+\sin\theta'\sin\theta=\cos\phi\cos\phi'+\sin\phi\sin\phi'$. But by properties of the circle, AE . AE′ = AF . AF′, whence $\cos\theta\cos\theta'=\cos\phi\cos\phi'$; therefore $\sin\theta\sin\theta'=\sin\phi\sin\phi'$; but $\sin\theta : \sin\phi=m:n$, and $\sin\theta' : \sin\phi'=m':n'$; therefore $mm'=nn'=ll'$, by symmetry.

This proves that a circle passes through the six points which are the feet of the perpendiculars from (l, m, n), (l^{-1}, m^{-1}, n^{-1}).

Again, let $(\alpha_1, \beta_1, 0)$, $(\alpha_2, \beta_2, 0)$ be the coordinates of F, F′;

then $\qquad \alpha_1=\text{OD}+\text{OF}\sin\text{B}$, and $\beta_1=\text{OE}+\text{OF}\sin\text{A}$,

therefore $\qquad \dfrac{\alpha_1}{\beta_1}=\dfrac{l+n\sin\text{B}}{m+n\sin\text{A}}$, and similarly $\dfrac{\alpha_2}{\beta_2}=\dfrac{l^{-1}+n^{-1}\sin\text{B}}{m^{-1}+n^{-1}\sin\text{A}}$.

The quadratic which has these roots is

$$(m+n\sin\text{A})(m^{-1}+n^{-1}\sin\text{A})\,\alpha^2+(l+n\sin\text{B})(l^{-1}+n^{-1}\sin\text{B})\,\beta^2$$
$$-\left\{(l+n\sin\text{B})(m^{-1}+n^{-1}\sin\text{A})+(l^{-1}+n^{-1}\sin\text{B})(m+n\sin\text{A})\right\}\alpha\beta=0.$$

To find the equation to the circle required, we have only to make this equation homogeneous by adding terms containing γ.

[The theorem follows at once from the consideration that (l, m, n), (l^{-1}, m^{-1}, n^{-1}) are the foci of a conic inscribed in the triangle, and the circle in question is that drawn on the major axis as diameter, since this circle is the locus of the feet of the perpendiculars from the foci on the tangents.]

1818. (Proposed by the EDITOR.)—Two points being taken at random within (1) a circle, or (2) a sphere, find the probability that the chord drawn through them is less than a given line.

Solution by PROFESSOR SYLVESTER.

1. Let 1 be the radius of the given circle, $2k$ the length of the given line, $k'^2 = 1-k^2$, and p the probability that the chord drawn through two arbitrary points within the circle does not exceed $2k$.

Divide the circle into concentric rings each of varying radius r. Then the arrangements out of which are to be sought the favourable cases, become a reduplication of rings of density $2\pi r dr$, combined respectively with circles of radius r. Let 2λ be the length of a chord in any such circle, which produced both ways to meet the *given* circumference generates a chord $2k$; also let A be the area of the smaller segment of the circle to radius r cut off by the chord λ. Then $\lambda^2 = r^2 - k'^2$; and so long as $r < k'$ every produced chord *exceeds* $2k$; but when $r > k'$, we have

$$A = \int_0^{\lambda} \frac{2\rho^2 d\rho}{(r^2-\rho^2)^{\frac{1}{2}}}, \quad \pi^2 p = \int_{k'}^{1} 4\pi r dr\,(2A); \text{ or if } \rho = rt,\ 1 - \frac{k'^2}{r^2} = u^2,$$

$$\pi p = 4\int_{k'}^{1} 4r^3 dr \int_0^u \frac{t^2 dt}{(1-t^2)^{\frac{1}{2}}} = 4k'^4 \int_0^k \delta_k \left(\frac{1}{1-k^2}\right)^2 dk \int_0^k \frac{k^2 dk}{(1-k^2)^{\frac{1}{2}}}$$

$$= 2\left(\sin^{-1} k - kk'\right) - 4k'^4 \int_0^k \frac{k^2 dk}{(1-k^2)^{\frac{5}{2}}} = 2\sin^{-1} k - 2kk' - \tfrac{4}{3}k^3 k'.$$

Thus, for example, the chance of the chord not exceeding the *radius* is $\frac{1}{3} - \frac{7\sqrt{3}}{12\pi}$, that is ·0117, or a little less than $\frac{1}{85}$.

2. The result is striking for simplicity when for a circular we substitute a spherical contour; for then we have

$$\left(\frac{4}{3}\pi\right)^2 p = 2\int_{k'}^{1} 4\pi r^2 dr \left(\frac{4\pi\lambda^4}{3r}\right) = \frac{32}{3}\pi^2 \int_{k'}^{1} r dr\,(r^2 - k'^2)^{\frac{3}{2}} = \frac{16}{9}\pi^2 k',$$

therefore in this case $p = k^6$.

Thus, for example, it is rather more than an even chance that a chord drawn through two points in the interior of a sphere shall *exceed eight-ninths* of the diameter, and exactly 63 against 1 that it shall exceed the *radius*.

[If we consider every possible position of the second point, whether nearer to or farther from the centre than the first, it is easy to see that the chord will not exceed $2k$ if the second point fall within an area S, consisting of two mixtilinear triangles bounded by the circumference of the given circle and two chords, each equal to $2k$, drawn through the first point; thus we find

$$S = 2\left(\cos^{-1}\frac{k'}{r} - k'\lambda\right);$$

therefore $\pi^2 p = \int_{k'}^{1} 2\pi r dr\,(S) = 2\pi \left(r^2 \cos^{-1}\frac{k'}{r} - k'\lambda - \tfrac{2}{3}k'\lambda^3\right)_{r=k'}^{r=1};$

whence we obtain $p = \frac{1}{\pi}(2\sin^{-1}k - 2kk' - \frac{4}{3}k^3k')$, as found in Professor SYLVESTER's Solution.

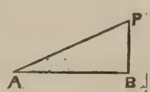
For the sphere we have only to suppose the circle with its connected lines to revolve round a diameter through the first point, then S generates a volume V within which the second point must lie in order that the chord through the two points may not exceed $2k$; and we find

$$V = \tfrac{4}{3}\pi\frac{k^3\lambda}{r}; \;\therefore\; (\tfrac{4}{3}\pi)^2 p = \int_{k'}^{1} 4\pi r^2 dr\,(V) = \left(\tfrac{16}{9}\pi^2k^3\lambda^3\right)_{r=k'}^{r=1};$$

whence we obtain $p = k^6$ as found in Professor SYLVESTER's Solution.]

1824. (Proposed by J. M. WILSON, M.A.)—Assuming that the bowler can run with any velocity less than v, and the batter can hit with any velocity less than u, and all less velocities and all directions (along the ground only) are equally probable, find the chance that the bowler will be able to stop a hit of the batter.

Solution by the PROPOSER.



Let the ball be struck at A with a velocity x in the direction AP, and let the bowler meet it at P by running in the direction BP, making an angle ϕ with AB, with a velocity y. Then $y = \frac{\sin\theta}{\sin\phi}x$, and therefore y is least when ϕ is a right angle for given values of x and θ, and $\sin\theta = \frac{y}{x}$. Hence the bowler can stop all balls (1) whose velocity is less than v, whether before or behind the wicket; (2) whose angle of direction θ is less than $\sin^{-1}\frac{v}{x}$.

The probability required is, therefore,

$$\frac{v}{u} + \int_v^u \frac{1}{\pi}\sin^{-1}\frac{v}{x}\cdot\frac{dx}{u} = \frac{v}{2u} + \frac{1}{\pi}\left\{\sin^{-1}\frac{v}{u} + \frac{v}{u}\log_e\frac{u+\sqrt{(u^2-v^2)}}{v}\right\}.$$

1840. (Proposed by Professor SYLVESTER.)—If, when L, M, N are three collinear points, [LMN] denote $+1$ or -1 according as M is within or external to the segment LN; prove the following theorems of four collinear points A, B, C, D, anyhow situated relatively to one another:—

(1).. [ACD][CAB] = [BDC][DBA] = −[ADC][DAB] = −[BCD][CBA];

(2).. $AC^2 + BD^2 - AD^2 - BC^2 = 2$ [ACD] [CAB] AB . CD;

(3).. [ACD] [CAB] AB . CD + [ADB] [DAC] AC . DB
+ [ABC] [BAD] AD . BC = 0.

Solution by W. H. LAVERTY.

The sign of [LMN] is the same as that of $\left[\frac{LM}{MN}\right]$, for the latter is positive or negative according as M is within or external to the segment LN; hence, substituting this last notation, we have

$$(\alpha)\quad [ACD]\,[CAB] = \left[\frac{AC}{CD}\right]\left[-\frac{AC}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];\ \text{since } AC^2 \text{ is positive.}$$

$$(\beta)\quad [BDC]\,[DBA] = \left[-\frac{BD}{CD}\right]\left[\frac{BD}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];$$

$$(\gamma)\quad -[ADC]\,[DAB] = -\left[-\frac{AD}{CD}\right]\left[-\frac{AD}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];$$

$$(\delta)\quad -[BCD]\,[CBA] = -\left[\frac{BC}{CD}\right]\left[\frac{BC}{AB}\right] = \left[-\frac{1}{AB\,.\,CD}\right];$$

$$(\epsilon)\quad [ADB]\,[DAC] = \left[\frac{AD}{DB}\right]\left[-\frac{AD}{AC}\right] = \left[-\frac{1}{AC\,.\,DB}\right];$$

$$(\zeta)\quad [ABC]\,[BAD] = \left[\frac{AB}{BC}\right]\left[-\frac{AB}{AD}\right] = \left[-\frac{1}{AD\,.\,BC}\right];$$

hence we see that (α), (β), (γ), (δ), are all equal; which proves (1).

Again, $AC^2 + BD^2 - AD^2 - BC^2 = -2\,.\,AB\,.\,CD$, as we should find by substituting $(AB+BC)$ for AC, &c.; also

$$2\,[ACD]\,[CAB]\,AB\,.\,CD = 2\left[-\frac{1}{AB\,.\,CD}\right] \times AB\,.\,CD = -2\,.\,AB\,.\,CD;$$

which proves (2).

Lastly, substituting from (α), (ϵ), (ζ), in (3), we have

$$-AB\,.\,CD - AC\,.\,DB - AD\,.\,BC = -AB\,.\,CD + (AB+BC)\,(BC+CD) - (AB+BC+CD)\,BC = 0;$$ which proves (3).

1837. (Proposed by J. GRIFFITHS, M.A.)—P is any point in the plane of a circle (C); Q any point on the polar of P with respect to (C); show that (C) cuts orthogonally the circle on PQ as diameter.

Solution by ARCHER STANLEY; R. WARREN, B.A.; W. H. LAVERTY; REV. J. L. KITCHIN, M.A.; *and others.*

The line joining the centre of the given circle to P cuts the polar of the latter in a point P_1 which is inverse to P, relative to (C). Again, PP_1Q being a right angle, the circle on PQ as diameter passes through the inverse points P and P_1, and consequently cuts (C) orthogonally. (Townsend's *Modern Geometry*, Vol. I., Art. 156.)

1842. (Proposed by R. Ball, M.A.)—Find the condition connecting the coefficients of two binary quartics, in order that there may be an arrangement of their roots $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ which will make

$$\begin{vmatrix} \alpha\alpha', & \alpha, & \alpha', & 1 \\ \beta\beta', & \beta, & \beta', & 1 \\ \gamma\gamma', & \gamma, & \gamma', & 1 \\ \delta\delta', & \delta, & \delta', & 1 \end{vmatrix} = 0.$$

Solution by the Proposer; R. Warren, B.A.; *and others.*

Let $(a, b, c, d, e)(x, 1)^4 = 0$; put $x = \dfrac{l+my}{\lambda+\mu y}$, and suppose for y we have $(a', b', c', d', e')(y, 1)^4 = 0$; then $I' = M^4 I$, $J' = M^6 J$, where I, J are the invariants of the quartics, and M the modulus of transformation. Let $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ be the corresponding roots of the two equations; then from the relations $\alpha = \dfrac{l+m\alpha'}{\lambda+\mu\alpha'}$, &c., we obtain

$$\mu\alpha\alpha' + \lambda\alpha - m\alpha' - l = 0, \quad \mu\beta\beta' + \lambda\beta - m\beta' - l = 0, \quad \&c.\ \&c.;$$

therefore, by eliminating $(\mu, \lambda, -m, -l)$, whatever l, m, λ, μ may be, we shall have for *one* arrangement that given in the Question, provided only that we have the relation $\dfrac{I'^3}{J'^2} = \dfrac{I^3}{J^2}$, obtained by eliminating M.

1864. (Proposed by the Rev. J. Blissard.)—Prove that

$$(1)\ldots 1 - n + \frac{n(n-1)}{1.2}\ldots\ \pm \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} = \pm \frac{(n-1)(n-2)\ldots(n-r)}{1.2\ldots r},$$

$$(2)\ldots \frac{1}{m+1}\ldots + \frac{1}{m+n} = \frac{n(n+1)\ldots(n+m)}{1.2\ldots m}\left\{\frac{1}{(m+1)^2} - \frac{n-1}{1}\cdot\frac{1}{(m+2)^2} + \&c.\right\}$$

Solution by Samuel Roberts, M.A.

Since $(1-x)^{n-1} = (1-x)^n(1+x+\ldots+x^r) + x^{r+1}(1-x)^{n-1}$, we get the first result by equating coefficients of x^r. Also

$$1 = \text{Co. of } x^{n-1} \text{ in } (1+x)^{n-1} = \text{Co. of } x^{n-1} \text{ in } (1+x)^{m+n}(1+x)^{-(m+1)}$$

$$= \frac{\Gamma(m+n+1)}{\Gamma(n)\,\Gamma(m+2)} - (m+1)\frac{\Gamma(m+n+1)}{\Gamma(n-1)\,\Gamma(m+3)} + \ldots$$

$$= \frac{\Gamma(m+n+1)}{\Gamma(n)\,\Gamma(m+1)}\left\{\frac{1}{m+1} - \frac{n-1}{1}\cdot\frac{1}{m+2} + \ldots\right\},$$

therefore $$\frac{\Gamma(m+1)}{\Gamma(m+n+1)} = \frac{1}{\Gamma(n)}\left\{\frac{1}{m+1} - \frac{n-1}{1}\cdot\frac{1}{m+2} + \ldots\right\};$$

and differentiating with regard to m, and writing $\dfrac{(m+n)(m+n-1)\ldots n}{1.2\ldots m}$

for $\frac{(m+n)\,..\,(m+1)}{1\,.\,2\,..\,(n-1)}$, we have the second result, which however is more directly obtained by use of $\int_0^1 z^m (1-z)^{n-1}\,dz = \frac{\Gamma(n)\,\Gamma(m+1)}{\Gamma(m+n+1)}$.

1810. (Proposed by C. TAYLOR, M.A.)—Prove that the value of the expression $\{\cos(\alpha-\beta)-\cos(\gamma-\delta)\}^2+\{\cos(\alpha+\beta)-\cos(\gamma+\delta)\}^2+\{\cos(\alpha+\gamma)\cos(\beta-\delta)-\cos(\beta+\delta)\cos(\alpha-\gamma)\}^2$ is unaltered by the interchange of β, γ.

Solution by J. WALMSLEY.

Writing a^2, b^2, c^2 for the terms of the given expression, and C for the sum of certain quantities not affected in value by the interchange of β, γ, we have

$$a^2+b^2 = 4\sin^2\tfrac{1}{2}(\alpha-\beta+\gamma-\delta)\sin^2\tfrac{1}{2}(\alpha-\beta-\gamma+\delta) + 4\sin^2\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\sin^2\tfrac{1}{2}(\alpha+\beta-\gamma-\delta)$$

$$= 2\sin^2\tfrac{1}{2}(\alpha-\beta-\gamma+\delta)\{1-\cos(\alpha-\delta)\cos(\beta-\gamma)-\sin(\alpha-\delta)\sin(\beta-\gamma)\}$$
$$+2\sin^2\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\{1-\cos(\alpha-\delta)\cos(\beta-\gamma)+\sin(\alpha-\delta)\sin(\beta-\gamma)\}$$
$$= C+2\sin(\alpha-\delta)\sin(\beta-\gamma)\sin(\alpha+\delta)\sin(\beta+\gamma);$$

$$c^2 = \tfrac{1}{4}\{\cos(\alpha+\beta+\gamma-\delta)+\cos(\alpha-\beta+\gamma+\delta)-\cos(\alpha+\beta-\gamma+\delta)$$
$$-\cos(\alpha-\beta-\gamma-\delta)^2\} = \{\sin(\alpha+\delta)\sin(\beta-\gamma)-\sin(\alpha-\delta)\sin(\beta+\gamma)\}^2;$$

therefore the given expression becomes

$$C+\sin^2(\alpha+\delta)\sin^2(\beta-\gamma)+\sin^2(\alpha-\delta)\sin^2(\beta+\gamma),$$

and is hence not altered in value by the proposed interchange.

1822. (Proposed by W. S. BURNSIDE, B.A.)—Prove that the area of a triangle circumscribing a conic is $ab^{-2}p_1p_2p_3$, where p_1 is one of the four perpendiculars from the vertex (1) on the focal vectors to the points of contact of tangents from the same vertex.

Solution by W. H. LAVERTY; R. WARREN, B.A.; J. DALE; *and others.*

Let a, b be the semi-axes of the conic, and α, β, γ the eccentric angles of the points of contact of the sides 23, 31, 12, respectively; then it may be readily shown that the lengths of the perpendiculars on the focal vectors to these points, drawn from the opposite vertices 1, 2, 3, are, respectively,

$$p_1 = b\tan\tfrac{1}{2}(\beta-\gamma),\quad p_2 = b\tan\tfrac{1}{2}(\gamma-\alpha),\quad p_3 = b\tan\tfrac{1}{2}(\alpha-\beta).$$

But (Salmon's *Conics*, Art. 231, ex. 9) the area of the triangle 123 is

$$ab\tan\tfrac{1}{2}(\alpha-\beta)\tan\tfrac{1}{2}(\beta-\gamma)\tan\tfrac{1}{2}(\gamma-\alpha),\ \text{or}\ ab^{-2}p_1p_2p_3.$$

Note on the Problems in regard to a Conic defined by five conditions of intersection. By Professor Cayley.

I use the word "intersection" rather than "contact," because it extends to the case of a 1-pointic intersection, which cannot be termed a contact. The conditions referred to are that the conic shall have with a given curve, at a point given or not given, a 1-pointic intersection, a 2-pointic intersection (= ordinary contact), a 3-pointic intersection, &c., as the case may be. It may be noticed that when the point on the curve is a given point, the condition of a k-pointic intersection is really only the condition that the conic shall pass through k given points; though from the circumstance that these are consecutive points on a conic, the formulæ for a conic passing through k discrete points require material alteration; for instance, in the two questions to find the equation of a conic passing through five given points, and to find, the equation of a conic having at a given point of a given curve 5-pointic intersection with the curve, the forms of the solutions are very different from each other.

The foregoing remark shows, however, that it is proper to detach the conditions which relate to intersections at given points; and consequently attending only to the conditions which relate to intersection at an unascertained point (of course the intersections referred to must be at least 2-pointic, for otherwise there is no condition at all) we may consider the conics which pass through four points and satisfy one condition; or which pass through three points and satisfy two conditions; or which pass through two points and satisfy three conditions; or which pass through one point and satisfy four conditions; or which satisfy five conditions. Considering in particular the last case, let 1 denote that the conic has 2-pointic intersection, 2 that it has 3-pointic intersection, ... 5 that it has 6-pointic intersection with a given curve at an unascertained point.

Then the problems are in the first instance

5 ; 4, 1 ; 3, 2; 3, 1, 1 ; 2, 2, 1; 2, 1, 1, 1 ; 1, 1, 1, 1, 1.

But the intersections may be intersections with the same given curve or with different given curves; and we have thus in all 27 problems viz., these are as given in the following table, where the colons (:) separate those conditions which refer to different curves:—

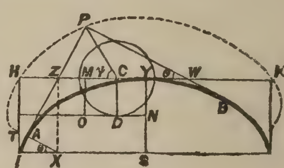
No. of Prob.	Conditions.	No. of Prob.	Conditions.	No. of Prob.	Conditions.
1	5	10	3, 1 : 1	19	3 : 1 : 1
2	4, 1	11	3 : 1, 1	20	2 : 2 : 1
3	3, 2	12	2, 2 : 1	21	2, 1 : 1 : 1
4	3, 1, 1	13	2, 1 : 2	22	2 : 1, 1 : 1
5	2, 2, 1	14	2, 1, 1 : 1	23	1, 1, 1 : 1 : 1
6	2, 1, 1, 1	15	2, 1 : 1, 1	24	1, 1 : 1, 1 : 1
7	1, 1, 1, 1, 1	16	2 : 1, 1, 1	25	2 : 1 : 1 : 1
8	4 : 1	17	1, 1, 1, 1 : 1	26	1, 1 : 1 : 1 : 1
9	3 : 2	18	1, 1, 1 ; 1, 1	27	1 : 1 : 1 ; 1 : 1

Thus Problem 1 is to find a conic having 6-pointic intersection with a given curve; Problem 2 a conic having 5-pointic intersection and also 2-pointic intersection with a given curve. .... Problem 7 is to find a conic having five 2-pointic intersections with (touching at five distinct points) a given curve. .... Problem 27 is to find a conic having 2-pointic intersection with

(touching) each of five given curves. Or we may in each case take the problem to be merely to find the number of the conics which satisfy the required conditions. This number is known in Prob. 1, for the case of a curve of the order m without singularities, viz., the number is $= m(12m-27)$. It is also known in Problems 25 and 26 in the case where the first curve (that to which the symbol 2, or 1, 1 relates) is a curve without singularities; and it is known in Prob. 27, viz., if m, n, p, q, r be the orders and M, N, P, Q, R the classes of the five curves respectively, then the number is $= (\mathrm{M}, m)(\mathrm{N}, n)(\mathrm{P}, p)(\mathrm{Q}, q)(\mathrm{R}, r)\{1, 2, 4, 4, 2, 1\}$, that is, $1\,\mathrm{MNPQR} + 2\,\Sigma \mathrm{MNPQ}r + \&c.$ The number is not, I believe, known in any other of the problems. In particular, (Prob. 7) we do not as yet know the number of the conics which touch a given curve at five points. It would be interesting to obtain this number; but (judging from the analogous question of finding the double tangents of a curve) the problem is probably a very difficult one.

1694. (Proposed by M. W. Crofton, B.A.)—Show that if tangents at right angles are drawn to a cycloid, the locus of their intersection is a trochoid, whose generating circle is equal to that of the cycloid (radius r) and rolls along the line which bisects the height of the cycloid at right angles; the distance of the describing point from the centre being $\frac{1}{2}\pi r$. Show how the *loops* of this trochoid form part of the locus.

Solution by the Proposer; J. Dale; *and others.*



Lemma.—It is easily proved that, if PA, PB be two perpendicular tangents to a cycloid, the portion ZW which they intercept on HK is always equal to πr.

Bisect now ZW in C, then $\mathrm{CP} = \frac{1}{2}\pi r$; draw the normal AX, then XZ is parallel to YS, the height. Bisect HY in M, then $\frac{1}{2}\pi r = \mathrm{HM} = \mathrm{CZ}$; hence $\mathrm{MC} = \mathrm{HZ} = \mathrm{IX} = 2r\theta$: but PCZ (or ψ) $= 2\theta$, therefore $\mathrm{MC} = r\psi$.

Describe a circle from C as centre with radius r; it touches ON in D (N being the middle point of YS); now, supposing this circle to have rolled along the line OD, from O, the angle through which it has turned is $\frac{\mathrm{OD}}{r} = \frac{\mathrm{MC}}{r} = \psi$; hence, if we suppose the line MH to have been rigidly connected with it in its first position, when its centre was at M, it will now have carried the point H from its original position to P; for $\mathrm{CP} = \mathrm{MH}$. Hence the locus sought is a trochoid THPK. The *loops* of this trochoid, below H and K, will be found to be the loci of the intersections of tangents to the cycloid with perpendicular tangents to the preceding and succeeding cycloids.

1778. (Proposed by W. S. BURNSIDE, B.A.)—If the line given by the equation $\alpha x + \beta y + \gamma z = 0$ intersect the conic $(a, b, c, f, g, h)(x, y, z)^2 = 0$ in the points P, Q; the tangents to the conic at these points meeting in the point R, and a focus being at the point F: prove that $\dfrac{\text{FP}\,.\,\text{FQ}}{\text{FR}^2} = \dfrac{\Pi^2}{\Pi^2 - \Theta\Sigma}$,

where $\Theta = (\text{A, B, C, F, G, H})(\sin \text{A}, \sin \text{B}, \sin \text{C})^2$,

$$\Sigma = (\text{A, B, C, F, G, H})(\alpha, \beta, \gamma)^2,\quad 2\Pi = \sin \text{A}\,\frac{d\Sigma}{d\alpha} + \sin \text{B}\,\frac{d\Sigma}{d\beta} + \sin \text{C}\,\frac{d\Sigma}{d\gamma}.$$

Solution by the PROPOSER.

1. If ϕ and ϕ_1 are the eccentric angles of the points P and Q; then $\tan^2 \frac{1}{2}(\phi - \phi_1) = -\Theta\Sigma\Pi^{-2}$. For if $\alpha x + \beta y + \gamma z$ and $\lambda \text{X} + \mu \text{Y} + \nu$ be identical, where $a\lambda = \cos \frac{1}{2}(\phi + \phi_1)$, $b\mu = \sin \frac{1}{2}(\phi + \phi_1)$, $\nu = -\cos \frac{1}{2}(\phi - \phi_1)$; we have $\Sigma \equiv q\,(a^2\lambda^2 + b^2\mu^2 - \nu^2) \equiv q \sin^2 \frac{1}{2}(\phi - \phi_1)$, whence $\Theta \equiv -q\text{M}$ (where $\text{M} = x \sin \text{A} + y \sin \text{B} + z \sin \text{C}$), and $\Pi \equiv -q\text{M}\nu \equiv q\text{M} \cos \frac{1}{2}(\phi - \phi_1)$; hence, finally, $-\Theta\Sigma\Pi^{-2} = \tan^2 \frac{1}{2}(\phi - \phi_1)$.

2. To prove that $\text{FP}\,.\,\text{FQ} = \text{FR}^2 \cos^2 \frac{1}{2}(\phi - \phi_1)$.

We have $\text{FP} = a(1 - e\cos\phi)$, and $\text{FQ} = a(1 - e\cos\phi_1)$;

whence $\text{FP}\,.\,\text{FQ} = a^2 \left\{\cos \frac{1}{2}(\phi + \phi_1) - e \cos \frac{1}{2}(\phi - \phi_1)\right\}^2 + b^2 \sin^2 \frac{1}{2}(\phi + \phi_1)$.

But the coordinates of R are $\dfrac{x}{a} = \dfrac{\cos \frac{1}{2}(\phi + \phi_1)}{\cos \frac{1}{2}(\phi - \phi_1)}$, $\dfrac{y}{b} = \dfrac{\sin \frac{1}{2}(\phi + \phi_1)}{\cos \frac{1}{2}(\phi - \phi_1)}$;

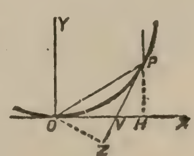
$$\therefore\ \text{FP}\,.\,\text{FQ} = \left\{(x - c)^2 + y^2\right\} \cos^2 \tfrac{1}{2}(\phi - \phi_1) = \text{FR}^2\,.\,\cos^2 \tfrac{1}{2}(\phi - \phi_1).$$

Hence $$\frac{\text{FP}\,.\,\text{FQ}}{\text{FR}^2} = \cos^2 \tfrac{1}{2}(\phi - \phi_1) = \frac{1}{1 + \tan^2 \frac{1}{2}(\phi - \phi_1)} = \frac{\Pi^2}{\Pi^2 - \Theta\Sigma}.$$

3. Hence if $\text{FP}\,.\,\text{FQ} : \text{FR}^2 = \text{F}_1\text{P}_1\,.\,\text{F}_1\text{Q}_1 : \text{F}_1\text{R}_1^2$ where two concentric conics are cut by the line $\alpha x + \beta y + \gamma z = 0$, the envelope of this line is the *parabola* $\Theta\Sigma' - \Theta'\Sigma = 0$. For the above relation gives $\Pi^2 : \Theta\Sigma = \Pi'^2 : \Theta'\Sigma'$, and since the conics are concentric, $\Pi : \Theta = \Pi' : \Theta'$.

1682. (Proposed by M. W. CROFTON, B.A.)—Find a curve such that the radii of curvature at any two points are to each other as the cubes of the tangents drawn from those points to meet each other.

Solution by the PROPOSER.



Refer the curve to a fixed tangent and normal, OX, OY; let a be the radius of curvature at O, ρ that at the point P;

then $$\frac{\rho}{a} = \frac{\text{PV}^3}{\text{OV}^3} = \frac{\text{PH}^3}{\text{OZ}^3} = \frac{y^3}{p^3};$$

whence we obtain $\dfrac{dr}{r^2} = a\,\dfrac{dp}{p^3}\sin^3\theta$;

or putting $u = \frac{1}{r}$, we have, as $\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2}$,

$$\frac{du}{d\theta} = a\frac{du}{d\theta}\sin^3\theta\left(u + \frac{d^2u}{d\theta^2}\right), \qquad \text{hence } \frac{d^2u}{d\theta^2} + u = \frac{1}{a\sin^3\theta},$$

the integral of which is $au = \sin\theta\int\frac{\cos\theta}{\sin^3\theta}\,d\theta - \cos\theta\int\frac{d\theta}{\sin^2\theta}$,

or
$$au = -\frac{1}{2\sin\theta} + \frac{\cos^2\theta}{\sin\theta} + C_1\sin\theta + C_2\cos\theta,$$

the equation of the curve sought: or, returning to rectangular coordinates,

$$ay = \tfrac{1}{2}x^2 + Ay^2 + Bxy,$$

A and B being arbitrary constants: hence, the only curves which possess the property are the conic sections.

[It may be shown as follows that in the conic $ax^2 + bxy + cy^2 = dy$ the radius of curvature at the origin is $\frac{d}{2a}$; for this equation may be written $a(x^2+y^2) - dy = (a-c)y^2 - bxy$; hence the circle $a(x^2+y^2) = dy$ cuts the conic on the lines $y = 0$, $(a-c)y = bx$, so that three of the intersections coincide at the origin; hence this is the osculating circle there.]

1826. (Proposed by G. O. Hanlon.)—The vertex of a triangle is fixed, while its base, of constant length, moves along a given line; show that the locus of the centre of the circumscribed circle is a parabola.

Solution by R. Warren, B.A.; *the* Rev. J. L. Kitchin, M.A.; J. Dale; W. H. Laverty; *the* Proposer; *and others.*

Take the fixed line on which the base moves as axis of y, and the perpendicular to it through the vertex as axis of x. Let $2c$ be the length of the base; and $(a, 0)$, (x, y) the respective coordinates of the vertex and the centre of the circumscribed circle.

Then obviously the equation of the required locus is

$$(a-x)^2 + y^2 = x^2 + c^2, \text{ or } y^2 = 2ax - (a^2 - c^2),$$

which is that of a parabola, whose parameter is $2a$ and whose vertex is situated on the axis of x at a distance of $\frac{a^2-c^2}{2a}$ from the origin.

[Another investigation of the locus is given in the *Note* to Quest. 1506, *Reprint*, Vol. II., p. 46; and the *envelope* of the series of circles is determined in the Solution to that Question.]

1827. (Proposed by W. H. LAVERTY.)—Find the values of x, y, z which make the function $u = f(x) \cdot \phi(y) \cdot \psi(z)$ a maximum; x, y, z being connected by the equation $a^{f(x)-\alpha} \cdot b^{\phi(y)-\beta} \cdot c^{\psi(z)-\gamma} = \text{A}$.

Solution by the REV. J. L. KITCHIN, M.A.; R. WARREN, B.A.; *the* PROPOSER; *and others.*

Differentiating function and condition, we have

$$\frac{f'(x)}{f(x)} \cdot dx + \frac{\phi'(y)}{\phi(y)} \cdot dy + \frac{\psi'(z)}{\psi(z)} \cdot dz = 0,$$

$$f'(x) \cdot dx \cdot \log a + \phi'(y) \cdot dy \cdot \log b + \psi'(z) \cdot dz \cdot \log c = 0.$$

Let $\dfrac{1}{f(x)} = k \cdot \log a$; $\therefore \dfrac{1}{\phi(y)} = k \cdot \log b$; and $\dfrac{1}{\psi(z)} = k \cdot \log c$,

$$\therefore \; 3 = k\left\{\log a^{f(x)} + \log b^{\phi(y)} + \log c^{\psi(z)}\right\} = k \log (\text{A} \cdot a^{\alpha} \cdot b^{\beta} \cdot c^{\gamma}),$$

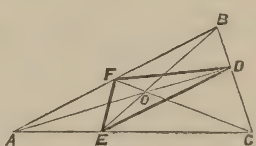
$$\therefore f(x) = \frac{\log (\text{A} \cdot a^{\alpha} \cdot b^{\beta} \cdot c^{\gamma})}{3 \log a}; \quad \phi(y) = \frac{\log (\text{A} \cdot a^{\alpha} \cdot b^{\beta} \cdot c^{\gamma})}{3 \log b}; \quad \psi(z) = \&\text{c.},$$

which give equations for determining x, y, z; also the maximum value is

$$u = f(x) \cdot \phi(y) \cdot \psi(z) = \frac{\left\{\log (\text{A} \cdot a^{\alpha} \cdot b^{\beta} \cdot c^{\gamma})\right\}^3}{\log a^3 \cdot \log b^3 \cdot \log c^3}.$$

1683. (Proposed by R. TUCKER, M.A.)—Of all the concurrently-connectant triangles that can be inscribed in a given triangle, prove that the maximum is that for which the concurrent point is the centroid of the given triangle. Find also the maximum triangle when the concurrent point lies on the perpendicular from one angle on the opposite side, or on the bisector of the angle; and find, furthermore, the locus of the concurrent point when the area of the connectant triangle is *constant.*

Solution by the PROPOSER; E. MCCORMICK; E. FITZGERALD; J. DALE; *and others.*



1. Take AC, AB, for axes, then the area of the triangle DEF, corresponding to the concurrent point O, or (x, y), will be

since d is $\left(\dfrac{bcx}{by+cx}, \dfrac{bcy}{by+cx}\right)$

e is $\left(\dfrac{cx}{c-y}, 0\right)$

f is $\left(0, \dfrac{by}{b-x}\right)$

$$bc \sin \text{A} \, (-u), \text{ where } u = \frac{xy\,(by+cx-bc)}{(b-x)\,(c-y)\,(by+cx)} \ldots\ldots\ldots\ldots (a).$$

Now $\dfrac{du}{dx} = \dfrac{b^2y^2(by+2cx-bc)}{(b-x)^2(c-y)(by+cx)^2}$, $\dfrac{du}{dy} = \dfrac{c^2x^2(cx+2by-bc)}{(b-x)(c-y)^2(by+cx)^2}$;
hence for a maximum or minimum

$$by+2cx = bc = cx+2by, \text{ or } x = \tfrac{1}{3}b,\ y = \tfrac{1}{3}c,$$

therefore O is the *centroid* of the triangle.

For the above point we find

$$\frac{d^2u}{dx^2} = \frac{27}{8b^2},\quad \frac{d^2u}{dy^2} = \frac{27}{8c^2},\quad \frac{d^2u}{dx\,dy} = \frac{27}{32bc};$$

whence we see that the triangle is a *maximum.*

2. Again, we have $\dfrac{dy}{dx} = -\dfrac{b^2y^2(c-y)\ (by+2cx-bc)}{c^2x^2(b-x)(cx+2by-bc)}$; hence when O is restricted to the perpendicular from A on BC, whose equation is $x\cos C = y\cos B$, we have

$$c^2x^2(b-x)(cx+by-bc)\cos C + b^2y^2(c-y)(by+2cx-bc)\cos B = 0,$$

or $$(a^2-bc\cos B\cos C)\,x^2 - 2(abc\cos B)\,x + b^2c^2\cos^2 B = 0\,;$$

$$\therefore\ x = \frac{bc\cos B}{a \pm (bc\cos B\cos C)^{\frac{1}{2}}};\qquad \text{maximum area} = \frac{bc^2\sin B\cos B\cos C}{a\pm 2(bc\cos B\cos C)^{\frac{1}{2}}}$$

3. In the case when O is restricted to the bisector of the angle A, we have, since the equation to this line is $y = x$,

$$b^2x^2(c-x)(bx+2cx-bc) + c^2x^2(b-x)(cx+2bx-bc) = 0,$$

or $$(b+c)(b^2+bc+c^2)\,x^2 - 2bc\,(b+c)^2\,x + b^2c^2(b+c) = 0,$$

whence $x = \dfrac{bc}{b\mp\sqrt{bc}+c}$; maximum area $= \dfrac{b^2c^2\sin A}{(b+c)(\sqrt{b}\mp\sqrt{c})^2}$.

4. From equation (α) above we see that if the triangle DEF is of constant area (viz. $k\Delta\,ABC = \frac{1}{2}kbc\sin A$) and the point O unrestricted in position, the locus of O will be a cubic given by the equation

$$2xy\,(by+cx-bc) + k\,(b-x)(c-y)(by+cx) = 0 \dots\dots\dots\dots (\beta).$$

If now we write $c = mb$, and arrange, we get

$$y^2 - m\,(b-x)\,y = -\frac{m^2bkx\,(b-x)}{2x-k\,(b-x)},$$

whence $$y = \frac{m\,(b-x)}{2} \pm \frac{m\,(b-x)^{\frac{1}{2}}\left\{2x(b-x)-k\,(b+x)^2\right\}^{\frac{1}{2}}}{2\left\{2x-k\,(b-x)\right\}^{\frac{1}{2}}}.$$

The asymptotes are given by the equations

$$y = \frac{kc}{2+k},\quad x = \frac{kb}{2+k},\quad y = -mx+\frac{2c}{2+k}.$$

The abscissas corresponding to the tangential ordinates of the *oval* will be given by the consideration that the two values of y must coincide, in which case we readily see that

$$2x\,(b-x) - k\,(b+x)^2 = 0, \text{ or } x = b\left\{\frac{1-k\pm(1-4k)^{\frac{1}{2}}}{2+k}\right\}.$$

From these values of x we see that when $k = \frac{1}{4}$ the oval becomes a point, viz. the centroid.

Again, if y_1, y_2 be ordinates corresponding to the same abscissa x, we have $y_1+y_2 = m(b-x)$, which shows that the oval cuts an ordinate produced to meet BC in such a way that the intercepts outside it are equal.

Since x may have any negative value or any positive value greater than b, we see that the curve is of the following nature:—

$k < \frac{1}{4}$, three infinite asymptotic branches passing through A, B, C, and an oval interior to the triangle ABC;
$k = \frac{1}{4}$, the oval reduces to the centroid;
$k > \frac{1}{4}$, three infinite asymptotic branches only.

The maximum and minimum ordinates correspond to the abscissas found from the equation $k(b-x)^2 + x(2x-b) = 0$, obtained by putting $\frac{dy}{dx} = 0$,

whence $$x = \frac{b}{2}\left\{\frac{1+2k \pm (1-4k)^{\frac{1}{2}}}{2+k}\right\}.$$

[*Otherwise:* $\frac{\triangle\,\text{OEF}}{\triangle\,\text{OBC}} = \frac{\text{OE}\,.\,\text{OF}}{\text{OB}\,.\,\text{OC}}$, $\frac{\text{OBC}}{\text{ABC}} = \frac{\text{OD}}{\text{AD}}$, $\therefore \frac{\text{OEF}}{\text{ABC}} = \frac{\text{OD}\,.\,\text{OE}\,.\,\text{OF}}{\text{OB}\,.\,\text{OC}\,.\,\text{AD}}$;

$$\therefore \frac{\text{DEF}}{\text{ABC}} = \frac{\text{OD}\,.\,\text{OE}\,.\,\text{OF}}{\text{OA}\,.\,\text{OB}\,.\,\text{OC}}\left(\frac{\text{OA}}{\text{AD}} + \frac{\text{OB}}{\text{BE}} + \frac{\text{OC}}{\text{CF}}\right) = 2\,\frac{\text{OD}\,.\,\text{OE}\,.\,\text{OF}}{\text{OA}\,.\,\text{OB}\,.\,\text{OC}}.$$

Let DEF $= k$, ABC $= 1$, BOC $= x$, COA $= y$, AOB $= z$; then we have

$$k = \frac{2xyz}{(y+z)(z+x)(x+y)} = \frac{2xyz}{yz+zx+xy-xyz};$$

therefore $1 + \frac{2}{k} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ ($= u$ suppose); also $x+y+z = 1$.

Now u is readily found to be a *minimum* ($u = 9$) when $x = y = z = \frac{1}{3}$, that is to say when O is the centroid of the triangle ABC; hence k, that is the triangle DEF, is then a *maximum* ($k = \frac{1}{4}$). When the area (k) of the triangle DEF is *constant*, the equation, in triangular coordinates (x, y, z) of the locus of O is the cubic

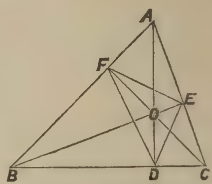
$$2xyz = k(y+z)(z+x)(x+y).$$

This cubic has been discussed at some length, with several illustrative figures for different values of the parameter, by Mr. WHITWORTH, in a very interesting paper in the *Messenger of Mathematics*, (Vol. II., pp. 123–127). The properties there investigated by *trilinear* coordinates agree with those which Mr. TUCKER has developed above by *Cartesian* coordinates: thus, for instance, the asymptotes are given by equations which we may interpret by saying that these lines are parallel to the sides of the given triangle ABC, and divide each of the sides which they cut so that the segment towards the parallel side is $\frac{1}{2}k$ times that towards the opposite angle.]

1782. (Proposed by J. WILSON.)—According as one of a triangle is a geometric, harmonic, or arithmetic mean between the other two, so is the cosine of one of the semi-angles of the *pedal triangle* a geometric, harmonic, or arithmetic mean between the cosines of the semi-angles of the other two.

Solution by E. CONOLLY; E. MCCORMICK; E. FITZGERALD; J. DALE; H. MURPHY; *and many others.*

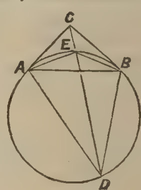
Let ABC be the triangle, DEF the *pedal triangle.* Then, since ABDE, ACDF are inscriptible in circles, the angle EDC = BAC = FDB, &c.; whence it follows that the semi-angles of the *pedal triangle* are the complements of the angles of the original triangle; the cosines of these semi-angles are, therefore, proportional to the sides on which their vertices lie. Whatever homogeneous relations, therefore, subsist between the sides of the original triangle, the same must subsist between the cosines of the semi-angles of the *pedal triangle.*



1762. (Proposed by J. O'CALLAGHAN.)—From the intersection of two tangents to a circle to draw a line, cutting the circumference in two points such that, if they are joined with the points of contact of the tangents, the rectangle contained by either pair of the opposite sides of the quadrilateral thus formed may be given or a maximum.

Solution by E. MCCORMICK; H. MURPHY; E. CONNOLLY; *and others.*

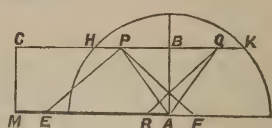
Let CA, CB be the two tangents, and CED the line required. Then, from the similarity of the triangles AEC and DAC, BEC and DBC, we have DA : AE = DC : CA (or CB) = DB : BE, therefore DA . BE = DB . AE = $\frac{1}{2}$AB . DE, by Ptolemy's theorem; and hence the solution of the problem is manifest.



1177. (Proposed by S. WATSON.)—Two rods, connected at their middle points by a string whose length exceeds half the sum of the lengths of the rods, are thrown at random on the ground; what is the chance that they will rest across each other?

Solution by the PROPOSER.

Let $2a$ be the length of the longer rod, $2b$ that of the shorter rod, and $2s$ ($>a+b$) the length of the string. Let AM represent half the longer rod, A being one end and M the middle of the rod. Draw BC parallel to AM, and AB, MC perpendicular



to AM. Take in CB any two points P, Q equidistant from B; draw PE = PF = QR = b, and join AP, AQ. Also with A as centre and radius b draw a circle cutting BC in H, K. Put $\angle$ EPF = 2θ, then AB = MC = $b \cos \theta$. Now $\angle$ AQR = APF, therefore EPA + AQR = EPF = 2θ. Hence we may always consider the middle of the shorter rod to lie between B and C, and to cross the longer rod while either end revolves through an angle 2θ. Hence doubting twice, first on account of the other half of the longer rod, and next because BC may lie on either side of AM, we obtain for the measure of the positions favourable for crossing

$$F = 16ab \int \theta d\,(b \cos\theta) = 16ab^2 \int_0^{\frac{1}{2}\pi} \theta \sin\theta \, d\theta = 16ab^2.$$

The measure of the whole number of positions is $W = (2\pi b)(4\pi s^2) = 8\pi^2 b s^2$: hence the required chance is

$$p = \frac{F}{W} = \frac{2ab}{\pi^2 s^2}; \text{ which, if } a = b = s, \text{ becomes } p = \frac{2}{\pi^2}.$$

[When one of the rods is half the length of the other, and the string half the length of the shorter rod, it is shown, in the *Lady's and Gentleman's Diary* for 1860, Quest. 1958, that the chance of crossing is $\frac{1}{2} + \frac{2}{\pi^2}$].

Solution of the Problem in Vol. II., p. 74, of the Reprint.

By Professor De Morgan.

A straight line being divided at random in two places; find the chance that the triangle formed by the thr e parts has all its angles acute.

The very peculiar method adopted by the Editor induced me to try whether a more direct method would offer any difficulty.

Let the whole line ($2a$) be filled with points: the number of points in a line being proportional to its length. When I say the number of points in a line is (as) x, I mean that it is $\infty \,.\, x$, the symbol ∞ being of one *value* in all lines. Thus dx contains $\infty \,.\, dx$ points, at *subequal* distances from the commencement of x. I call two magnitudes *subequal* which differ by an infinitely small part of either.

First, how many ways are there of dividing $2a$ into three parts? Answer, $\frac{1}{2} \infty \,.\, 2a\,(\infty \,.\, 2a - 1)$ or $\infty^2 \,.\, 2a^2$; say as $2a^2$. This may be verified by

$$\int_0^{2a} \int_x^{2a} dx\, dy.$$

Next, how many of these ways give triangles? Let x, y, z be the consecutive divisions of $2a$; let x be the greatest, and z the least. Six times the number of such divisions is the answer required. Now x must lie between $\frac{2}{3}a$ and a, since $y + z > x$; and y must lie between $\frac{1}{2}(2a - x)$ and x; and

$$\int_{\frac{2}{3}a}^{a} \int_{\frac{1}{2}(2a-x)}^{x} dx\, dy = \tfrac{1}{12}a^2.$$

Hence $\frac{1}{2}a^2$ is (as) the number of triangles; or the chance of getting a triangle is $\frac{1}{4}$.

The number of acute-angled triangles is six times the number of cases in which $x > y > z$, and $y^2 + z^2 > x^2$, or

$$y^2 + (2a - x - y)^2 > x^2, \text{ or } y^2 - (2a - x)\,y + 2a\,(a - x) > 0.$$

This expression $(x < a)$ has two positive roots, when x is great enough; their sum is $2a - x$, so we must take the greater for the lowest value of y.

It is
$$\tfrac{1}{2}\left\{2a - x + \sqrt{(2a + x)^2 - 8a^2}\right\} = t.$$

So long as the roots are imaginary, we must integrate dy from $\frac{1}{2}(2a - x)$ to x; but when the roots become real, we must integrate from t to x. This amounts to $\int_{\frac{2}{3}a}^{a}\int_{t}^{x} dx\,dy$, with the imaginary part of the result omitted.

Now
$$\int_t^x dy = \tfrac{1}{2}\left\{3x - 2a - \sqrt{(2a + x)^2 - 8a^2}\right\},$$

and
$$\int_{\frac{2}{3}a}^{a}\int_{t}^{x} dx\,dy = \left\{-\tfrac{2}{3} + \tfrac{2}{3}\sqrt{(-\tfrac{8}{9})} - 2\log\left(\tfrac{2}{3} + \sqrt{-\tfrac{1}{18}}\right)\right\}a^2.$$

Of this the real part is $(-\frac{2}{3} - \log\frac{1}{2})\,a^2$; and six times this is $(6\log 2 - 4)a^2$; which is (as) the number of acute-angled triangles. And $2a^2$ being (as) the whole number of triangles, the chance of an acute-angled triangle is

$$3\log 2 - 2.$$

This agrees with the result of the Editor, whose ingenuity I greatly admire: but I doubt if any soul alive would fully believe either of us, if it were not for the other.

The Problem—A line being divided at hazard into n parts, required the chance of a polygon—offers no difficulty of translation into a multiple integral; I should like to see the result of the integration.

[The Editor obtained the result at first by the aid of the Integral Calculus, and his Solution by this method was published in the *Educational Times* for November, 1859. In the next Number however (the Editorship being then in other hands) the correctness of this method and result was called in question by a correspondent, who substituted instead an erroneous solution, leading to a different result. The present Editor thereupon, in reply, drew up an article, in which, after pointing out the errors in the second Solution, he gave a geometrical solution in corroboration of the result obtained by his former method. This last solution was published in the *Educational Times* for January, 1860, and afterwards reproduced with some alterations in the volume of the *Reprint* referred to by Professor De Morgan. The foregoing remarks are offered here in explanation of the peculiarity of the method.

The Problem enunciated at the end of Professor De Morgan's Solution will be found proposed, with the *result* annexed, as Question 1878.]

1887. (Proposed by Professor SYLVESTER.)—Find the mean value of the volume of a tetrahedron, three of whose vertices lie respectively in three non-intersecting edges, and the fourth at the centre of a given parallelepiped.

Solution by the PROPOSER.

It may easily be proved that the proportion of the mean tetrahedron in question to the volume of the parallelepiped is independent of the form and dimensions of the latter. For greater simplicity, then, we may suppose it to be a cube whose sides are each 2. Take the principal axes of the cube through the centre as the coordinate axes: then, calling x, y, z the distances of the variable points from the centres of their respective ranges, the volume of the tetrahedron is $\frac{V}{6}$, where $V = \pm \begin{vmatrix} x, & 1, & -1 \\ -1, & y, & 1 \\ 1, & -1, & z \end{vmatrix} = \pm (xyz + x + y + z)$, with the understanding that V is to be always *positive.*

There will be 8 cases reduceable to a duplication of the four following; viz., x, y, z, all positive, or all but one positive.

When x, y, z are all positive, V, operated upon by

$$\int_0^1 dz \int_0^1 dy \int_0^1 dx \text{ gives } \tfrac{13}{8}, \text{ as is easily found.}$$

When x is negative, say $-\xi$, the positive value of V is $-\xi yz - \xi + y + z$ from $\xi = 0$ to $\xi = \frac{y+z}{yz+1}$ say (ξ), provided the latter is not greater than unity. This condition is necessarily fulfilled, for $1 - (\xi) = \frac{(1-y)(1-z)}{yz+1}$, which, since y, z are each included between 0, 1, is necessarily positive. From $\xi = (\xi)$ to $\xi = 1$ the positive value of V is $\xi yz + \xi - y - z$. Hence the derived integral may be found by operating upon this latter quantity with $\int_0^1 dz \int_0^1 dy \int_0^1 d\xi$, and adding the result of operating upon $-\xi yz - \xi + y + z$ with $2\int_0^1 dz \int_0^1 dy \int_0^{(\xi)} d\xi$. The first operate is $-\frac{3}{8}$. Again, the result of the operation $2\int_0^{(\xi)} d\xi$ upon its operand is

$$\frac{(y+z)^2}{yz+1} = \frac{y^2}{yz+1} + \frac{z^2}{yz+1} - \frac{2}{yz+1} + 2.$$

Hence the result of the complete operation will be $2 - 2P + 2Q$, where

$$P = \int_0^1 \int_0^1 dy\, dz\, \frac{1}{1+yz} = \int_0^1 dy\, \frac{\log(1+y)}{y} = 1 - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{4^2} \dots = \frac{\pi^2}{12},$$

$$Q = \int_0^1 \int_0^1 dy\, dz\, \frac{y^2}{1+yz} = \int_1^2 d\omega\, (\omega - 1) \log \omega = \frac{1}{4}.$$

Thus $2 - 2P + 2Q = \frac{5}{2} - \frac{1}{6}\pi^2$; and the sum of the two complementary integrals is $\frac{17}{8} - \frac{1}{6}\pi^2$. Consequently the total aggregate $\Sigma(V dx dy dz)$ for

all eight cases combined $= 2 \times \frac{13}{8} + 6 \times \left(\frac{17}{8} - \frac{\pi^2}{6}\right) = 16 - \pi^2$. Hence the fractional part of the volume which represents the mean value required is

$$\frac{16-\pi^2}{6 \times 8}, \text{ or } \frac{1}{3} - \frac{\pi^2}{48}.$$

1859. (Proposed by J. GRIFFITHS, M.A.)—Show that the locus of the centres of equilateral hyperbolas touching the sides of a given obtuse-angled triangle is the self-conjugate circle of this triangle.

I. *Solution by* ARCHER STANLEY.

Let us first consider the system of conics inscribed in a given triangle ABC, and dividing harmonically a given rectilinear segment DE. In this system there will be but two conics which touch DE, since it is only in D or in E that two harmonic conjugates relative to D, E can coincide with one another, and only one inscribed conic can be drawn to touch DE at a given point. From this it follows that D and E are the only poles of DE, relative to the several conics of the system, which lie in DE; in other words, the *locus of all such poles is a conic* (P) *passing through* D *and* E.

If the line AD cut BC in a, then the line connecting A with a (either directly or through infinity) will represent a flattened conic (ellipse or hyperbola) included in the system; and the pole of DE, relative to it, will be the harmonic conjugate d of D relative to A, a. Hence d will be on (P). Similarly e, the harmonic conjugate of E relative to A and the intersection of AE and BC, will be another point on (P), whence it follows that BC will be the polar of A relative to (P). In a similar manner, CA is the polar of B, and AB of C; in other words ABC *is self-conjugate relative to* (P).

Allow D and E to represent the imaginary circular points at infinity, and every conic of the system will become an equilateral hyperbola, every pole of DE will be the centre of a hyperbola, and the locus (P) of this centre will be a circle relative to which ABC is a self-conjugate triangle.

II. *Solution by* S. ROBERTS, M.A.; J. DALE; E. MCCORMICK: *and others.*

Taking the given triangle as that of reference, the condition necessary in order that the general inscribed conic should be an equilateral hyperbola is

$$l^2 + m^2 + n^2 + 2mn \cos A + 2nl \cos B + 2lm \cos C = 0;$$

we have also, if (α, β, γ) be the coordinates of the *centre*,

$$l = \sin A\,(-\alpha \sin A + \beta \sin B + \gamma \sin C), \text{ \&c., \&c.};$$

hence finally the required locus is, by substitution,

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0,$$

representing therefore the circle with respect to which the given triangle is self-conjugate.

[See Ferrers' *Trilinear Coordinates*, pp. 83, 42, 31.]

1857. (Proposed by Professor CAYLEY.)—If for shortness we put

$$P = x^3+y^3+z^3,\quad Q = yz^2+y^2z+zx^2+z^2x+xy^2+x^2y,\quad R = xyz,$$
$$P_0 = a^3+b^3+c^3,\quad Q_0 = bc^2+b^2c+ca^2+c^2a+ab^2+a^2b,\quad R_0 = abc\ ;$$

then (α, β, γ) being arbitrary, show that the cubic curves $\begin{vmatrix} \alpha, & \beta, & \gamma \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = 0$ pass all of them through the same nine points, lying six of them upon a conic and three of them upon a line; and find the equations of the conic and line, and the coordinates of the nine points of intersection; find also the values of $(\alpha : \beta : \gamma)$ in order that the cubic curve may break up into the conic and line.

Solution by T. COTTERILL, M.A.; PROFESSOR CREMONA; S. ROBERTS, M.A.; W. A. WHITWORTH, M.A.; J. DALE; *and others.*

The determinant vanishes, if the corresponding constituents of the second and third rows coincide, which is evidently the case for the coordinates of the six points formed from the permutations of (abc). It also vanishes, if each constituent of the second row becomes 0, or by the values

$$(x=0,\ y+z=0),\quad (y=0,\ z+x=0),\quad (z=0,\ x+y=0).$$

Let $\qquad S = x+y+z,\quad U = x^2+y^2+z^2,\quad V = yz+zx+xy\ ;$

also $\qquad S_0 = a+b+c,\quad U_0 = a^2+b^2+c^2,\quad V_0 = bc+ca+ab\ ;$

then the six points lie on the conic $UV_0-VU_0 = 0$, and the three points lie on the line $S = 0$.

In the determinant, α, β may be changed into $\alpha+\beta$, $\beta+3\gamma$, when P and Q will become $P+Q$, $Q+3R$, which are both divisible by S.

Hence making $\alpha = -\beta = 3\gamma$, we have

$$\begin{vmatrix} 3, & -3, & 1 \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = \begin{vmatrix} P+Q, & Q+3R \\ P_0+Q_0, & Q_0+3R_0 \end{vmatrix} = SS_0\,(UV_0-VU_0).$$

1730. (Proposed by Professor CAYLEY.) — Show that (I) the condition in order that the roots k_1, k_2, k_3 of the equation

$$\gamma k^3+(-g-\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma)\,k^2+(-g-\tfrac{3}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma)\,k-\alpha = 0\ \ldots\ldots(A)$$

may be connected by a relation of the form $k_3(k_1-k_2)-(k_2-k_3)=0\,\ldots(1)$; and (II) the result of the elimination of a, b, c from the equations

$$a^2(b+c) = -2\alpha\ldots(2),\quad b^2(c+a) = 2\beta\ldots(3),\quad c^2(a+b) = -2\gamma\ldots(4),$$
$$(b-c)(c-a)(a-b) = -4g\ldots(5)\ ;$$

are each $4(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g^3+4(-\Sigma\alpha^3\beta+4\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma)\,g^2$
$+(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,g+2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = 0\ldots\ldots\ldots\ldots(B).$

Solution by SAMUEL BILLS.

I. Here, in addition to the relation (1), we have from (A)

$$k_1+k_2+k_3 = \gamma^{-1}(g+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta-\tfrac{3}{2}\gamma) = m \text{ (say)}\ \ldots\ldots\ldots(6),$$

$$k_2k_3 + k_3k_1 + k_1k_2 = -\gamma^{-1}(g + \tfrac{3}{2}\alpha + \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = n \text{ (say)} \dots\dots (7),$$
$$k_3^3 - mk_3^2 + nk_3 - a = 0 \dots\dots\dots\dots (8).$$

From (1), and (6), $k_1 = \dfrac{m + (m-2)\ k_3 - k_3^2}{1 + 2k_3}$, $k_2 = \dfrac{k_3(m+1-k_3)}{1+2k_3}$.

Substituting these results in (7) we obtain

$$3k_3^4 - (2m-3)\,k_3^3 - (m^2 + 2m - 3 - 4n)\,k_3^2 - (m^2 + 2m - 4n)\,k_3 + n = 0 \ \dots (9).$$

From (9) $-(3k_3 + m + 3)$ (8) we have

$$(m+n+3)\,k_3^2 - (m^2 + 2m + mn - n - 3\alpha)\,k_3 + (m+3)\,\alpha + n = 0 \ \dots\dots (10).$$

Now (see Hirsch's *Algebra*, pp. 118, 119) the result of eliminating x from the equations $p + qx + rx^2 = 0$, $p_1 + q_1x + r_1x^2 + s_1x^3 = 0$, is

$$p^3s_1^2 + p^2rr_1^2 + pr^2q_1^2 + r^3p_1^2 - qr^2p_1q_1 + (q^2 - 2pr)\,(rp_1r_1 + pq_1s_1)$$
$$+ (3pqr - q^3)\,p_1s_1 - pqrq_1r_1 - p^2qr_1s_1 = 0 \dots\dots\dots (C).$$

Substituting from (8) and (10) in (C), restoring the values of m and n, and arranging the result, we obtain the relation (B). It is unnecessary to give the *details* of this last step, which are easily supplied.

II. Putting $bc + ca + ab = t$, we have from (2) $-$ (3), &c., &c.,

$$(a-b)\,t = 2(\beta - \alpha),\ (b-c)\,t = 2(\gamma - \beta),\ (c-a)\,t = 2(\alpha - \gamma);$$

therefore by (5), $gt^3 = 2(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$, which determines t;

also $a - b = 2t^{-1}(\beta - \alpha) = u$ (say), and $b - c = 2t^{-1}(\gamma - \beta) = v$ (say);

therefore $a = b + u$ and $c = b - v$; hence substituting in (2) and (4),

we have $(b+u)^2(2b-v) = -2\alpha$, and $(b-v)^2(2b+u) = -2\gamma$;

or $$2b^3 + (4u - v)\,b^2 + 2u(u-v)\,b - u^2v + 2\alpha = 0 \dots\dots\dots\dots (11),$$

and $$2b^3 - (4v - u)\,b^2 - 2v(u-v)\,b + uv^2 + 2\gamma = 0 \dots\dots\dots\dots (12).$$

From $\{(11) - (12)\} \div (u+v)$, remembering that $u + v = 2t^{-1}(\gamma - \alpha)$,

we have $$3b^2 + 2(u-v)\,b - uv - t = 0 \dots\dots\dots\dots (13).$$

Restoring the values of t, u, v, and substituting from (11) and (13) in (C), we shall obtain the relation (B) as before.

1834. (Proposed by Professor Cayley.)—1. It is required to find on a given cubic curve three points A, B, C, such that, writing $x=0$, $y=0$, $z=0$ for the equations of the lines BC, CA, AB respectively, the cubic curve may be transformable into itself by the inverse substitution $(\alpha x^{-1}, \beta y^{-1}, \gamma z^{-1})$ in place of x, y, z respectively, α, β, γ being disposable constants.

2. In the cubic curve $ax(y^2 + z^2) + by(z^2 + x^2) + cz(x^2 + y^2) + 2lxyz = 0$ the inverse points (x, y, z) and (x^{-1}, y^{-1}, z^{-1}) are corresponding points (that is, the tangents at these two points meet on the curve).

I. *Solution by the* Proposer; S. Roberts, M.A.; *and others.*

Since the points A, B, C are on the curve, the equation is of the form

$$fy^2z + gz^2x + hx^2y + iyz^2 + jzx^2 + kxy^2 + 2lxyz = 0.$$

Hence this equation must be equivalent to

$$\frac{f\beta^2\gamma}{y^2z}+\frac{g\gamma^2\alpha}{z^2x}+\frac{h\alpha^2\beta}{x^2y}+\frac{i\beta\gamma^2}{yz^2}+\frac{j\gamma\alpha^2}{zx^2}+\frac{k\alpha\beta^2}{xy^2}+\frac{2l\alpha\beta\gamma}{xyz}=0,$$

$$\text{or,}\quad j\frac{\alpha}{\beta}y^2z+k\frac{\beta}{\gamma}z^2x+i\frac{\gamma}{\alpha}x^2y+h\frac{\alpha}{\gamma}yz^2+f\frac{\beta}{\alpha}zx^2+g\frac{\gamma}{\beta}xy^2+2lxyz=0,$$

which will be the case if

$$f=j\frac{\alpha}{\beta},\ g=k\frac{\beta}{\gamma},\ h=i\frac{\gamma}{\alpha},\ i=h\frac{\alpha}{\gamma},\ j=f\frac{\beta}{\alpha},\ k=g\frac{\gamma}{\beta}.$$

This implies $fgh=ijk$; and if this condition be satisfied, then $\alpha:\beta:\gamma$ can be determined, viz., we have $\alpha:\beta:\gamma=if:ij:hf$, which satisfy the remaining equations, so that the only condition is $fgh=ijk$.

Writing in the equation of the curve $x=0$, we find $fy^2z+iyz^2=0$, that is, the line $x=0$ meets the curve in the points $(x=0,\ y=0)$, $(x=0,\ z=0)$, and $(x=0,\ fy+iz=0)$. We have thus on the curve the three points

$$(x=0,\ fy+iz=0),\ (y=0,\ gz+jx=0),\ (z=0,\ hx+ky=0),$$

and in virtue of the assumed relation $fgh=ijk$, these three points lie in a line. Hence the points A, B, C must be such that BC, CA, AB respectively meet the curve in points A′, B′, C′ which three points lie in a line; that is, we have a quadrilateral whereof the six angles A, B, C, A′, B′, C′ all lie on the curve. It is well known that the opposite angles A and A′, B and B′, C and C′ must be *corresponding points*, that is, points the tangents at which meet on the curve. And conversely taking A, C any two points on the curve, A′ a corresponding point to A (any one of the four corresponding points), then AC, A′C will meet the curve in the corresponding points B′, B; and AB, A′B′ will meet on the curve in a point C′ corresponding to C, giving the inscribed quadrilateral (A, B, C, A′, B′, C′); the triangle ABC is therefore constructed.

It is to be remarked that the equation $fgh=ijk$ being satisfied, we may without any real loss of generality write $f=j$, $g=k$, $h=i$, and therefore $\alpha=\beta=\gamma$; hence changing the constants we have the theorem: the inverse points (x, y, z), (x^{-1}, y^{-1}, z^{-1}) are corresponding points on the curve

$$ax(y^2+z^2)+by(z^2+x^2)+cx(x^2+y^2)+2lxyz=0.$$

II. *Solution by* PROFESSOR CREMONA; J. DALE; T. COTTERILL, M.A.; *and others.*

On sait que la cubique donnée H_3 admet généralement trois systèmes de coniques tangentes en trois points (*Teoria geom. delle curve piane*, 150). Chacun de ces systèmes est formé par les poloconiques des droites du plan par rapport à une cubique C_3, dont H_3 est la Hessienne. Soit R une droite (fixe) qui coupe H_3 in $a'b'c'$; la poloconique de R (par rapport à C_3) touchera H_3 en trois points *abc correspondants* (dans le sens que M. Cayley donne à ce mot) à $a'b'c'$ ($abca'b'c'$ sont les sommets d'un quadrilatère complet). Toute conique Σ circonscrite au triangle *abc* est (*Teoria*, 137) la poloconique mixte de R et d'une autre droite S; et réciproquement une droite quelconque du plan détermine, avec R, une poloconique mixte, qui passe toujours par *abc*. Ainsi, les coniques par *abc* et les droites du plan constituent deux réseaux projectifs. Un point quelconque *m*, considéré comme commun à un faisceaux de droites (de coniques par *abc*) détermine un point *m′*, commun aux coniques par *abc* (aux droites) correspondantes.

L'un quelconque des deux points homologues mm' est le pole de R par rapport à la conique polaire de l'autre (relative à C_3). Donc, nous avons une *transformation quadrique*, où les quatre points doubles sont les poles de R par rapport à C_3: ces poles forment un quadrangle complet dont le triangle diagonal est abc (*Teoria*, 134).

Si m est un point de H_3, la conique polaire de m est une couple de droites dont le point de croisement est correspondant à m; donc ce point est m'. C'est-à-dire que la cubique H_3 se transforme en soi-même ; et deux points homologues se *correspondent* entre eux.

On satisfait donc à la question de M. Cayley en prenant sur H_3 trois points abc où cette courbe soit touchée par une même conique K_2. Soit $\alpha yz+\beta zx+\gamma xy=0$ l'équation de K_2, et $axy^2+a'xz^2+byz^2+b'yx^2+czx^2+c'zy^2+2hxyz=0$ l'équation d'une cubique quelconque par abc. Cette cubique sera tangente à K_2 en abc si l'on a $a=m\gamma$, $a'=n\beta$, $b=n\alpha$, $b'=l\gamma$, $c=l\beta$, $c'=m\alpha$; l, m, n trois quantités arbitraires. Et la substitution $(x, y, z)=\left(\dfrac{\alpha}{lx}, \dfrac{\beta}{my}, \dfrac{\gamma}{nz}\right)$ transforme la cubique en soi-même.

En prenant $l=\alpha$, $m=\beta$, $n=\gamma$, on a la deuxième partie de la question proposée.

1871. (Proposed by Professor MANNHEIM.)—The envelope of a circle whose diameter is a chord, fixed in direction, of a given conic, is another conic whose foci are at the extremities of that diameter of the former which is conjugate to the fixed direction. Prove this, and find where the circle touches its envelope.

I. *Solution by* ARTHUR COHEN, B.A. ; *and* J. H. TAYLOR, B.A.

Take as axis of x that diameter of the conic which bisects the chords of the given direction, and as axis of y the diameter perpendicular to the former. Let $2a$ be the length of the former diameter, and $2b$ the length of the conjugate diameter. Then if $2r$ be the length of any one of the chords, the coördinates of whose middle point are $(h, 0)$, we have evidently

$$\frac{r^2}{(a+h)(a-h)}=\frac{b^2}{a^2}, \text{ or } r^2=b^2-\frac{b^2}{a^2}h^2;$$

hence the equation to the circle on such chord as diameter is

$$(x-h)^2+y^2=r^2, \text{ or } h^2\left(1+\frac{b^2}{a^2}\right)-2hx+x^2+y^2-b^2=0 \quad \ldots\ldots\ldots (1).$$

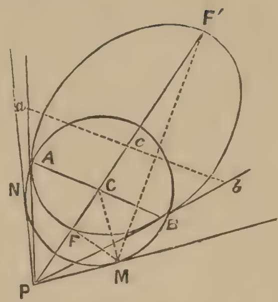
The envelope of such circles has for its equation (Salmon's *Conics*, Art. 283)

$$\left(1+\frac{b^2}{a^2}\right)(x^2+y^2-b^2)=x^2, \text{ or } \frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1 \quad \ldots\ldots\ldots (2),$$

which is evidently a conic whose foci are at $(0, +a)$ and $(0, -a)$, or the extremities of the diameter conjugate to the given chords.

Comparing (1) with (2) we readily see that the abscissa x of the point where the envelope touches the circle described on the chord whose abscissa is h is given by the equation $x=h\left(1+\dfrac{b^2}{a^2}\right)$.

II. *Solution by* Archer Stanley.



Let AB be a chord, given in direction, of the ellipse FAF′B. Let C be the middle point, and P the pole of AB, so that PC, cutting the ellipse in F and F′, is the diameter conjugate to the given direction. Now, if PM and PN be the tangents from P to the circle (C) described on AB as diameter, M and N will be the points in which (C) touches its envelope. For if a parallel to AB cut the tangents PA, PB produced in a and b it is clear that the circle (c) described on ab as diameter will likewise touch PM and PN, since P will be the centre of similitude of (C) and (c), and as ab approaches AB, the circle (c) approaches to coincidence with the circle described on the chord of the ellipse which is adjacent to AB. PM and PN, therefore, are tangents at M and N to the envelope of (C).

Now, the pencil M (P, F, C, F′) being manifestly harmonic, and the conjugate rays MP, MC perpendicular to one another, the latter bisects the angles formed by MF and MF′; but the tangent to the envelope at every point M thereon being equally inclined to the connectors of M with two fixed points F, F′, we at once conclude that the envelope is a conic of which F and F′ are the foci.

III. *Solution by the* Rev. R. Townsend, M.A.

The corresponding property in Geometry of Surfaces, viz.—“The envelope of a sphere whose diameter is a variable chord, fixed in direction, of a given quadric, is another quadric, of which the section of the original by the diametral plane conjugate to the fixed direction is a focal conic,”—presents no greater difficulty; for, if it be true in the particular case when the fixed direction is perpendicular to a principal plane of the surface, it is manifestly true in every case.

But in that case, the equation of the original quadric being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

if α and β be the variable coordinates of the middle point of the chord, supposed perpendicular to the plane of xy, that of the variable sphere is evidently

$$(x-\alpha)^2+(y-\beta)^2+z^2=c^2\left(1-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}\right),$$

between which and the relations

$$(x-\alpha)=\frac{c^2}{a^2}\alpha,\quad (y-\beta)=\frac{c^2}{b^2}\beta,$$

obtained from it immediately by the ordinary process for the determination of envelopes, the elimination of α and β gives that of the required envelope, viz.,

$$\frac{x^2}{a^2+c^2}+\frac{y^2}{b^2+c^2}+\frac{z^2}{c^2}=1,$$

which is therefore as above stated.

It follows, of course, from the above, conversely, that "If a variable sphere have in every position double contact with a fixed quadric, the chord of double contact being always parallel to the same axis of the surface, the extremities of its diameter parallel to the chord describe another quadric, having double contact with the original at the extremities of the axis, and passing through its focal conic in the principal plane perpendicular to the axis." The latter part of this is evident *à priori* from the known property, (see Salmon's *Geometry of Three Dimensions*, 2nd ed., Art. 138,) that every focus of a quadric is an evanescent sphere having double contact with the surface.

1861. (Proposed by C. TAYLOR, M.A.) — Prove that the difference between the sum of the sines and the sum of the cosines is greater or less than unity according as the triangle is acute or obtuse-angled.

Solution by SAMUEL ROBERTS, M.A.

The difference in question may be written

$$D = \sin A + \sin B + \sin(A+B) - \cos A - \cos B + \cos(A+B),$$

and the maxima or minima are given by

$$\sin A + \cos A = \sin B + \cos B = \sin C + \cos C, \quad \text{or} \quad \sin 2A = \sin 2B = \sin 2C,$$

with the condition $A + B + C = 180°$. We have therefore the systems

$$A,\ B,\ C \text{ not } > 90° \ldots\ldots \text{max. } (60°, 60°, 60°), \text{ min. } (90°, 90°, 0);$$

$$A \text{ or } B \text{ or } C \text{ not } < 90° \ldots\ldots \text{max. } (90°, 90°, 0), \text{ min. } (0, 0, 180°).$$

Hence in an acute triangle, the difference in question lies between $\frac{3}{2}(\sqrt{3}-1)$ and 1, or is >1; and in an obtuse triangle, the value lies between 1 and -1, or is <1.

II. *Solution by* J. DALE; REV. J. L. KITCHIN, M.A.; *and others.*

Let A be the greatest angle; then $D >$ or < 1 according as

$$2\cos\tfrac{1}{2}(A-B)(\cos\tfrac{1}{2}C - \sin\tfrac{1}{2}C) > \text{or} < 1 + \cos C - \sin C$$

$$> \text{or} < 2\cos\tfrac{1}{2}C(\cos\tfrac{1}{2}C - \sin\tfrac{1}{2}C)$$

or as $\frac{1}{2}(A-B) <$ or $> \frac{1}{2}C$, or as $A <$ or $> \frac{1}{2}(A+B+C) <$ or $> 90°$.

Note sur l'Intégration des Equations Différentielles Simultanées et Linéaires. Par E. PROUHET.

La méthode d'Ampère pour la résolution des équations

$$\left.\begin{aligned} A\frac{dy}{dx} + B\frac{dz}{dx} + Cy + Dz = E \\ A'\frac{dy}{dx} + B'\frac{dz}{dx} + C'y + D'z = E' \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (I)$$

exige que l'on ramène ces deux équations à deux autres dont l'une ne contienne que $\frac{dy}{dx}$ et l'autre que $\frac{dz}{dx}$. On peut éviter cette opération préalable en procédant de la manière suivante.

Je suppose en premier lieu que les coefficients A, B, C, D, A′, B′, C′, D′, soient constants. J'ajoute les deux équations (I) après les avoir multipliées respectivement par 1 et par la constante θ. J'obtiens ainsi

$$\frac{d}{dx}\left\{(A+A'\theta)y+(B+B'\theta)z\right\} + (C+C'\theta)y + (D+D'\theta)z = E+E'\theta \;..\; (1).$$

Je pose $(A+A'\theta)\,y+(B+B'\theta)\,z = u\ldots.(2)$, $\frac{C+C'\theta}{A+A'\theta} = \frac{D+D'\theta}{B+B'\theta} = k..(3)$;

et l'équation (1) se réduit à l'équation linéaire

$$\frac{du}{dx} + ku = E+E'\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

De la première des équations (3) on tire deux valeurs constantes (θ_1, θ_2) de l'inconnue θ. Soient u_1 et u_2 les valeurs correspondantes de u, obtenus en integrant l'équation (4). On aura

$$(A-A'\theta_1)\,y+(B+B'\theta_1)\,z = u_1, \quad (A+A'\theta_2)\,y+(B+B'\theta_2)\,z = u_2;$$

d'où l'on déduira les valeurs des fonctions inconnues y et z.

Quand les coefficients des premiers membres des équations (I) sont des fonctions de x, on procède d'une façon analogue mais en considérant θ comme une fonction de x. Dans ce cas le premier membre de l'équation (1) doit être diminué de

$$\left(\frac{dA}{dx} + \frac{dA'}{dx}\theta + A'\frac{d\theta}{dx}\right)y + \left(\frac{dB}{dx} + \frac{dB'}{dx}\theta + B'\frac{d\theta}{dx}\right)z$$

termes qu'il a fallu ajouter pour compléter la dérivée de $(A+A'\theta)\,y + (B+B'\theta)\,z$ ou de u. L'équation qui donne les valeurs de θ est alors

$$\frac{C+C'\theta-\frac{dA}{dx}-\frac{dA'}{dx}\theta - A\frac{d\theta}{dx}}{A+A'\theta} = \frac{D+D'\theta-\frac{dB}{dx}-\frac{dB'}{dx}\theta - B'\frac{d\theta}{dx}}{B+B'\theta}\ldots\ldots(5).$$

Cette équation est du premier ordre, mais non linéaire. Quand on saura l'intégrer, on aura deux valeurs de θ en attribuant à la constante arbitraire deux valeurs distinctes, et le calcul s'achèvera comme plus haut. On peut simplifier l'équation (5) en supposant que A et A′ sont deux constants, de qui est toujours permis.

La méthode précédente s'étend facilement au cas de trois équations simultanées. Nous n'examinerons que le cas où les coefficients seront constants.

Soient

$$\left.\begin{aligned}
A\frac{dy}{dx} + B\frac{dz}{dx} + C\frac{dt}{dx} + Dy + Ez + Ft &= G \\
A'\frac{dy}{dx} + B'\frac{dz}{dx} + C'\frac{dt}{dx} + D'y + E'z + Ft &= G' \\
A''\frac{dy}{dx} + B''\frac{dz}{dx} + C''\frac{dt}{dx} + D''y + E''z + F''t &= G''
\end{aligned}\right\} \ldots\ldots\ldots\ldots (\text{I}').$$

J'ajoute ces équations respectivement multipliées par les constantes 1, θ, λ.

J'aurai $\frac{d}{dx}\left\{(A+A'\theta+A''\lambda)\,y+(B+B'\theta+B''\lambda)\,z+(C+C'\theta+C''\lambda)\,t\right\}$

$$+(D+D'\theta+D''\lambda)\,y+(E+E'\theta+E''\lambda)\,z+(F+F'\theta+F'\lambda)\,t$$
$$= G+G'\theta+G''\lambda \ldots\ldots (1').$$

Je pose $(A+A'\theta+A''\lambda)\,y+(B+B'\theta+B''\lambda)\,z+(C+C'\theta+C''\lambda)\,t = u \,.. (2')$,

$$\frac{D+D'\theta+D''\lambda}{A+A'\theta+A''\lambda} = \frac{E+E'\theta+E''\lambda}{B+B'\theta+B''\lambda} = \frac{F+F'\theta+F''\lambda}{C+C'\theta+C''\lambda} = k \;\ldots\ldots\ldots (3'),$$

et l'équation (1′) se réduit à l'équation linéaire

$$\frac{du}{dx} + ku = G+G'\theta+G''\lambda \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4').$$

Les équations (3′) donnent, tout calcul fait, trois valeurs de θ et trois valeurs correspondantes de λ. L'intégration de l'équation (4′) donnera trois valeurs correspondantes de u. Si l'on substitue à θ, λ, u, dans l'équation (2′) tour à tour θ_1, λ_1, u_1; θ_2, λ_2, u_2; θ_3, λ_3, u_3; on aura trois équations pour déterminer les fonctions inconnues y, z, t.

1638. (Proposed by W. K. Clifford.)—Find the condition that the general equation of the third order may represent a cubic whose asymptotes form an equilateral triangle; and show that this is always the case when the curve passes through three points and their three pairs of antifoci.

Solution by the Proposer.

Three lines forming an equilateral triangle meet the line at infinity in a point-cubic whose Hessian is the circular points. Now let

$$(a, b, c, d)\,(x, y)^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the terms of the highest order in the general equation of the third degree in Cartesian coordinates; then the three lines represented by (1) are parallel to the asymptotes. Now the Hessian of (1) is $(ac-b^2)\,x^2+(ad-bc)\,xy+(bd-c^2)\,y^2$; and in order that this may be identical with x^2+y^2 we must have $(ad=bc,\ ac-b^2=bd-c^2)$, which are the conditions required.

To prove the proposition, let A, B, C be the three points, and P, Q the circular points at infinity. Let the equation to the three lines PA, PB, PC be $U=0$, and to the three lines QA, QB, QC, $V=0$. Then the nine intersections of the cubics U, V are the three points and their three pairs of antifoci. Any other cubic through those intersections may be represented by $U=kV$. Let U', V' be the terms of highest order in U, V; then

$U'-kV'$ will be the terms of highest order in $U-kV$. But U', V' must be perfect cubes, representing the circular points; say x^3, y^3. Then $U'-kV'$ is x^3-ky^3. But the Hessian of x^3-ky^3 is $-kxy$. That is, every cubic represented by $U=kV$ meets the line at infinity in a point-cubic whose Hessian is the circular points. Or, which is the same thing, the asymptotes of every such cubic form an equilateral triangle.

There is no difficulty in finding the conditions when the equation is given in a homogeneous form. We substitute for z, from the equation of the line at infinity, in the cubic and in any circle; let the former substitution give (1), and the latter, $Ax^2+2Bxy+Cy^2=0$; then the conditions are

$$\frac{ac-b^2}{A}=\frac{ad-bc}{2B}=\frac{bd-c^2}{C}.$$

1825. (Proposed by H. R. Greer, B.A.)—Prove that, in space, the locus of a point such that, if perpendiculars be drawn from it to the faces of a tetrahedron, their feet shall lie in a plane, is the surface

$$\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}=0,$$

A, B, C, D representing the areas of the faces, and x, y, z, w the perpendiculars drawn on them from any point.

Solution by W. Allen Whitworth, M.A.

Let $$l\alpha+m\beta+n\gamma+r\delta=0 \ldots\ldots\ldots\ldots \text{(i.)}$$

be the equation to the plane containing the feet of the perpendiculars from (x, y, z, w) on the faces A, B, C, D; then, if cos AB, cos BC, &c. denote the *external* angles between the planes A and B, B and C, &c., so that, by projection,

$$A+B\cos AB+C\cos AC+D\cos AD=0 \text{ (and similar equations)} \ldots \text{(ii.)},$$

the equations to the perpendicular from (x, y, z, w) on A may be written

$$\frac{\alpha-x}{1}=\frac{\beta-y}{\cos AB}=\frac{\gamma-z}{\cos AC}=\frac{\delta-w}{\cos AD}=\rho,$$

and since this line meets the plane (i.) on the plane $\alpha=0$, we get

$$m(y-x\cos AB)+n(z-x\cos AC)+r(w-x\cos AD)=0.$$

We get three similar equations from considering the other perpendiculars. Therefore, eliminating $l:m:n:r$, we get

$$\begin{vmatrix} 0, & y-x\cos AB, & z-x\cos AC, & w-x\cos AD \\ x-y\cos BA, & 0, & z-y\cos BC, & w-y\cos BD \\ x-z\cos CA, & y-z\cos CB, & 0, & w-z\cos CD \\ x-w\cos DA, & y-w\cos DB, & z-w\cos DC, & 0 \end{vmatrix}=0,$$

which, in virtue of (ii.), becomes

$$\begin{vmatrix} Ax+By+Cz+Dw, & \&c. & \\ Ax+By+Cz+Dw, & \&c. & \left\{\begin{matrix}\text{the other columns}\\ \text{as before.}\end{matrix}\right. \\ Ax+By+Cz+Dw, & \&c. & \\ Ax+By+Cz+Dw, & \&c. & \end{vmatrix}=0,$$

or $$\begin{vmatrix} \frac{1}{x}, & \frac{1}{x}-\frac{1}{y}\cos AB, & \frac{1}{x}-\frac{1}{z}\cos AC, & \frac{1}{x}-\frac{1}{w}\cos AD \\ \frac{1}{y}, & 0 \quad , & \&c. & \&c. \\ \frac{1}{z}, & \frac{1}{z}-\frac{1}{y}\cos BC, & \&c. & \&c. \\ \frac{1}{w}, & \frac{1}{w}-\frac{1}{y}\cos BD, & \&c. & \&c. \end{vmatrix} = 0,$$

or again, in virtue of (ii.),

$$\begin{vmatrix} \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}, & \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}, & \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}, & \frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w} \\ \frac{1}{y} \quad , & 0 \quad , & \frac{1}{y}-\frac{1}{z}\cos BC \;, & \frac{1}{y}-\frac{1}{w}\cos BD \\ \frac{1}{z} \quad , & \frac{1}{z}-\frac{1}{y}\cos BC, & 0 \quad , & \frac{1}{z}-\frac{1}{w}\cos CD \\ \frac{1}{w} \quad , & \frac{1}{w}-\frac{1}{y}\cos BD, & \frac{1}{w}-\frac{1}{z}\cos CD \;, & 0 \end{vmatrix} = 0,$$

or $$\left(\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}\right) \begin{vmatrix} 0 \quad , & \frac{1}{z}\cos BC, & \frac{1}{w}\cos BD \\ \frac{1}{y}\cos BC\,, & 0 \quad , & \frac{1}{w}\cos CD \\ \frac{1}{y}\cos BD, & \frac{1}{z}\cos CD, & 0 \end{vmatrix} = 0,$$

or $$\frac{1}{y}\cdot\frac{1}{z}\cdot\frac{1}{w}\left(\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w}\right) = 0;$$

hence $\frac{A}{x}+\frac{B}{y}+\frac{C}{z}+\frac{D}{w} = 0$ represents the locus required.

1867. (Proposed by W. H. LAVERTY.)—From a point O_1 on a conic, $(n-1)$ lines are drawn to points $O_2, O_3 \ldots O_n$ on the conic; $O_1O_3, O_1O_4 \ldots O_1O_n$ being inclined at angles $\alpha_3, \alpha_4 \ldots \alpha_n$ to O_1O_2; find the product of $O_1O_2 \,.\, O_1O_3 \ldots O_1O_n$, (1) in the general case, (2) when the conic becomes a circle, (3) when $O_1O_2O_3 \ldots O_n$ is a regular polygon in the circle.

Solution by S. ROBERTS, M.A.; REV. J. L. KITCHIN, M.A.; *the* PROPOSER; *and others.*

We may take O_1 as origin of rectangular coordinates. Then

$(A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha)\,\rho^2 + (D\cos\alpha + E\sin\alpha)\,\rho = 0$ gives

$$(1)\ldots \rho_1\rho_2\ldots\rho_{n-1}=(-)^{n-1}\frac{D}{A}\cdot\frac{D\cos\alpha_3+E\sin\alpha_3}{A\cos^2\alpha_3+B\sin\alpha_3\cos\alpha_3+C\sin^2\alpha_3}\cdots\cdots$$
$$\cdots\frac{D\cos\alpha_n+E\sin\alpha_n}{A\cos^2\alpha_n+B\sin\alpha_n\cos\alpha_n+C\sin^2\alpha_n};$$

$(2)\ldots \rho_1\rho_2\ldots\rho_{n-1}=(-)^{n-1}D\,.\,A^{1-n}\,.\,(D\cos\alpha_3+E\sin\alpha_3)\ldots(D\cos\alpha_n+\&c.)$;

(3).. since the side of a regular polygon of n sides is $2R\sin\frac{\pi}{n}$, we have

$$\rho_1\rho_2\ldots\rho_{n-1}=(-)^{n-1}\,2R\sin\frac{\pi}{n}\,.\,2R\sin\frac{2\pi}{n}\ldots.2R\sin\frac{(n-1)\pi}{n}.$$

But by Euler's formula we have

$$\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\ldots\sin\left\{\theta+\frac{(n-1)\pi}{n}\right\}=2^{1-n}\,.\,\sin n\theta,\quad\text{whence}$$

$$\sin\frac{\pi}{n}\,.\,\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}\,.=2^{1-n}\,.\,\frac{\sin n\theta}{\sin\left\{\theta+\frac{(n-1)\pi}{n}\right\}}\left(\theta=\frac{\pi}{n}\right)=2^{1-n}n;$$

therefore $\rho_1\rho_2\ldots\rho_{n-1}=nR^{n-1}$, a result immediately deducible also from Cotes' theorem, for we have $\rho_1\rho_2\ldots\rho_{n-1}=\dfrac{x^n-R^n}{x-R}\,[x=R]=nR^{n-1}$.

1699. (Proposed by W. GODWARD.)—Let O_1, O_2, O_3, be the centres of the escribed circles touching the sides BC, CA, AB, respectively, of the triangle ABC; and P_1, P_2, P_3 the feet of the perpendiculars from the vertices A, B, C on those sides; prove that O_1P_1, O_2P_2, O_3P_3 intersect in a point the sum of the trilinear coordinates of which is $\dfrac{R+r}{R-r}r$.

Solution by W. H. LAVERTY; J. DALE; E. FITZGERALD; *the* PROPOSER; *and others.*

Writing, for shortness' sake, (l, m, n) for $(\cos A, \cos B, \cos C)$ the trilinear equations of O_1P_1, O_2P_2, O_3P_3 are readily found to be

$$m\beta-n\gamma=(n-m)\,\alpha,\quad n\gamma-l\alpha=(l-n)\,\beta,\quad l\alpha-m\beta=(m-l)\,\gamma;$$

and since $l\,(\text{eq. }O_1P_1)+m\,(\text{eq. }O_2P_2)+n\,(\text{eq. }O_3P_3)\equiv 0$, these three lines meet in a point whose coordinates, found from any two of the equations, are

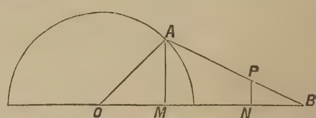
$$\frac{\alpha}{m+n-l}=\frac{\beta}{n+l-m}=\frac{\gamma}{l+m-n}=\frac{\Sigma(\alpha)}{\Sigma(\cos A)}=\frac{2\Delta}{\Sigma(a)-\Sigma(a\cos A)}$$
$$=\frac{2\Delta}{\frac{2\Delta}{r}-\frac{2\Delta}{R}}=\frac{Rr}{R-r};$$

$$\therefore\ \Sigma(\alpha) = \frac{r}{\mathrm{R}-r}\,\Sigma(\mathrm{R}\cos\mathrm{A}) = \frac{\mathrm{R}+r}{\mathrm{R}-r}\,.\,r.$$ (McDowell's *Exercises.* Prop. 96).

[It is clear that the concurrence of the lines in nowise depends on the values of l, m, n (see also the Note at the end of the Solution of Quest. 1726): and the sum of the coordinates of the centre of homology of the triangles $O_1O_2O_3$, $P_1P_2P_3$, for *any* values of l, m, n, is $\Sigma(\alpha) = \frac{(l+m+n)\,\Delta}{ls_1+ms_2+ns_3}$. If, for example, $l:m:n = a:b:c$, the result agrees with that given in the Note at the end of the Solution of Quest. 1616 (*Reprint*, Vol. III., p. 31.]

1703. (Proposed by C. BICKERDIKE.) — Given the length of the connecting rod of a horizontal steam engine, and the length of the stroke; find the locus of a given point in the connecting rod during one revolution of the crank.

Solution by J. DALE; E. FITZGERALD; E. McCORMICK; *and many others.*



Let OA = half the length of the stroke $= r$; AB (the connecting rod) $= a$; P any given point in AB; and PB $= b$. Then the question is to find the locus of P when A moves round the circle (O). Taking O as origin, OB as axis of x, and putting ON $= x$, PN $= y$, $\angle$AON $= \theta$, $\angle$PBN $= \phi$, we have $r\cos\theta + a\cos\phi = x + y\cot\phi$; also $\sin\phi = \frac{y}{b}$,

$$\sin\theta = \frac{a}{\gamma}\sin\phi = \frac{ay}{br},\ \cos\phi = \frac{1}{b}(b^2-y^2)^{\frac{1}{2}},\ \cos\theta = \frac{1}{br}(b^2r^2-a^2y^2)^{\frac{1}{2}};$$

hence, substituting these values in the above equation, we get, for the equation of the required locus,

$$(a-b)\ (b^2-y^2)^{\frac{1}{2}} + (b^2r^2-a^2y^2)^{\frac{1}{2}} = bx,$$

whence we see that the locus is a curve of the fourth order.

If $r=a$, the locus becomes $\frac{x^2}{(2a-b)^2} + \frac{y^2}{b^2} = 1$, which represents an ellipse whose semi-axes are $2a-b$ and b.

[When $r=a$, the *complete* locus consists of a semi-ellipse and a semicircle, having a common diameter perpendicular to OB through O; these *two* equations of the *second* degree being respectively obtained from Mr. DALE's equation of the *fourth* degree by taking the radicals therein of the *same* or of *different* signs. This will also be seen to be in conformity with the geometrical circumstances of the motion, the range of B being $2r$.]

On some extensions of the fundamental proposition in M. Chasles' Theory of Characteristics. By W. K. Clifford.

I mean by the "fundamental proposition" the following, viz. :—

"If a variable system of two points on a right line be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions; then there are $a+b$ points on the right line at which the system of two points coalesces into one point."

This principle has been admirably extended by Dr. Salmon to the case of two dimensions, thus :—"If a variable system of two points in a plane be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions, and that p pairs of points, each constituting a position of the system, may be found upon an arbitrary right line; then there are $a+b+p$ points in the plane at which the system of two points coalesces into one point."

The principle admits of further extension in two directions. First, we may consider a system of more than two points; and secondly, we may consider the system as subject to a less number of relations than is sufficient to determine a single point. We are thus led to the following propositions :—

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a surface of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a surface of order b, and so on; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a surface of order $a+b+\ldots.$

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a curve of order a, and when all but the second point are taken arbitrarily the second point is determined to lie on a curve of order b, and so on, and that when all but the first two points are taken arbitrarily there are on an arbitrary right line p pairs of points each constituting a position of the first two points, and that q, r, ... are the corresponding numbers for the other pairs of points of the system; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a curve of order $a+b+\ldots\ldots+p+q+r+\ldots.$

It is not worth while to state the analogous propositions for Geometry of one and two dimensions, or the correlative propositions for lines and planes. I go on to exemplify the application of these propositions.

Let us begin with Mr. Thomson's cubic. (Quest. 1545; *Reprint*, Vol. II., p. 57). A conic is inscribed in a triangle so that the normals at the points of contact meet in a point; it is required to find the locus of this point. Consider now a variable system of three points, subject only to this condition; that if perpendiculars be drawn from them respectively to the three sides of a triangle, a conic may be drawn touching the sides of the triangle at the feet of those perpendiculars. Then if we take two of the points arbitrarily, we determine two of the points of contact of the inscribed conic; that is, we determine the conic itself uniquely, and therefore the third point of contact; and the normal at this point is therefore the locus of the third point of the system. That is to say, we have a variable system of three points so related that when any two of the points are taken arbitrarily, the locus of the third is a straight line; consequently there are points in the plane at which the system of three points coalesces into one point (that is, where the

three normals meet in a point), and the locus of such points is a curve of order $(1+1+1=)\,3$.

To complicate the question, let us suppose that a conic is drawn to touch three given *conics*, so that the normals at the points of contact may meet in a point. Here, as before, we take our variable system of three points, one on each of the three normals. Take two of the points arbitrarily; from each of these we can draw four normals to the corresponding conic. Pairing these together, we have 16 pairs of points of contact. Now when we have given two tangents and their points of contact, the number of conics of the system which can be drawn to touch a given conic is 4. By determining two points of our variable system we have, therefore, determined 64 conics; on the third *given* conic these determine 64 points of contact, and the normals through these may be held to constitute a curve of the 64th order. Thus we have a variable system of three points so related that when any two of them are taken arbitrarily, the third is determined to lie on a curve of the 64th order; consequently the locus of those points at which the system coalesces into one point, or the three normals meet in a point, is a curve of the order $(3\times 64=)\,192$. More generally, if we substitute for the variable conic a curve of order m, class n, and deficiency D, or say a curve of *species* (m, n, D), and for the three fixed conics $\frac{1}{2}(m+n-\mathrm{D}+2)$ curves of orders $m_1, m_2, \ldots.$ and of classes $n_1, n_2, \ldots.$, then the corresponding locus will be of the order $3\phi(m, n, \mathrm{D}) \,.\, (m_1+n_1)\,(m_2+n_2)\ldots.$, where $\phi(m, n, \mathrm{D})$ is the number of curves of species (m, n, D) which can be drawn through $(m+n-\mathrm{D}+1)$ points, or touching $(m+n-\mathrm{D}+1)$ lines.

For another example, let us find the locus of those points the feet of the perpendiculars from which to four lines or planes in space are coplanar. In both these cases the locus comes out primarily of the fourth order; but the plane at infinity is evidently a part of the locus, the remainder of which is thus of the third order. In both cases the envelope of the plane through the feet of the perpendiculars is of the fourth class, and touches the plane at infinity. I conjecture that the imaginary circle is a curve of contact.

If a conicoid be drawn to touch five straight lines, so that the normal planes at the points of contact meet in a point, the locus is of this point of the tenth order. And so on *ad libitum*.

1854. (Proposed by Chief Justice COCKLE, F.R.S.) — Solve the differential equation $\dfrac{dy}{dx}+by^2=ax^2$; or, differential added to multiple of square of dependent variable equal to multiple of square of independent variable.

Solution by the PROPOSER.

The following discussion includes all cases of the equation

$$\frac{dy}{dx}+by^2=ax^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with the exception of that in which $m=1$. But in this excepted case the Solution may (by means of the investigations given at pp. 81, 82 of Hymers's

Differential Equations, 1839) be expressed by a definite integral. Hence it may be said that the equation (1) is in all cases whatever solvable either by definite or indefinite integrals.

Instead of (1) let us take as our starting point the equation

$$\frac{d^2y}{dx^2} + C^2\left\{(2n+1)\,x\right\}^{\frac{m-4n}{2n+1}} y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

to which (1) may always by appropriate transformations be reduced, and in which C, m, and n are arbitrary constants which are not supposed to be subject to any other conditions than that neither C nor $2n+1$ shall vanish. This being so, let

$$x = \frac{t^{2n+1}}{2n+1}, \quad \therefore \; \frac{dx}{dt} = t^{2n}, \quad \therefore \; \frac{dt}{dx}\cdot\frac{d^2x}{dt^2} = \frac{2n}{t} \ldots\ldots\ldots (3, 4, 5),$$

then, changing the independent variable from x to t, (2) becomes

$$\frac{d^2y}{dt^2} - \frac{2n}{t}\cdot\frac{dy}{dt} + C^2t^my = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now change the dependent variable, and let

$$y = t^nv, \quad \therefore \; \frac{dy}{dt} = t^n\frac{dv}{dt} + nt^{n-1}v \ldots\ldots\ldots\ldots\ldots (7, 8),$$

and
$$\frac{d^2y}{dt^2} = t^n\frac{d^2v}{dt^2} + 2nt^{n-1}\frac{dv}{dt} + n(n-1)\,t^{n-2}v \ldots\ldots\ldots\ldots (9).$$

Then, after substitution and reduction, (6) becomes

$$\frac{d^2v}{dt^2} + \left\{C^2t^m - (n^2+n)\frac{1}{t^2}\right\}v = 0 \ldots\ldots\ldots\ldots\ldots (10).$$

Let $m=0$, then, as we know (Hymers, ibid. pp. 83—85), the equation

$$\frac{d^2v}{dt^2} + \left\{C^2 - (n^2+n)\frac{1}{t^2}\right\}v = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

has for its solution $v = \beta t^{-n}\left(\frac{d^n}{d\xi^n}\right)_{\xi=C^2}\left\{\frac{\cos(t\sqrt{\xi}+\alpha)}{\sqrt{\xi}}\right\} \ldots\ldots\ldots (12)$

when n is an integer: while the same equation (11) has for its solution

$$v = \beta t^{n+1}\int_{-C}^{C}(\xi^2-C^2)^n\cos(t\xi+\alpha)\,d\xi \ldots\ldots\ldots\ldots\ldots (13)$$

when, in (10), n is a fraction; and in either solution α and β are arbitrary constants.

To solve the equation proposed in the Question, I put it under the successive forms

$$\frac{dy}{dx} + y^2 = ax^2, \text{ and } \frac{d^2y}{dx^2} + ax^2y = 0;$$

and then (remembering that the m in (2), which is an arbitrary quantity not necessarily identical with the m in (1), is to vanish) I put

$$2 = -\frac{4n}{2n+1}, \quad \text{or} \quad 4n+2 = -4n, \quad \text{or} \quad n = -\frac{1}{4} \ldots\ldots\ldots (14),$$

and hence obtain a reduced equation which, by properly assigning a, may be expressed by

$$\frac{d^2v}{dt^2} + \left\{1 - \left(\frac{1}{16} - \frac{1}{4}\right)\frac{1}{t^2}\right\} v = 0 \text{ .... (15),}$$

whereof the solution is (compare Hymers, ibid.)

$$v = \beta t^{\frac{3}{4}} \int_{-1}^{1} \frac{\cos(t\xi + \alpha)\, d\xi}{\sqrt[4]{(\xi^2 - 1)}} \text{ .... (16),}$$

and whereon that of the proposed equation depends. For any value of m, the index of the dexter of (1), we have, in place of (14), the conditions

$$m = -\frac{4n}{2n+1}, \quad \text{or } 2mn + m = -4n, \quad \text{or } n = -\frac{m}{2m+4} \text{ ...... (17),}$$

and so we may solve the general case. But the particular cases have certain relations one to the other which may be important as suggesting relations between the definite integrals which enter into them. These relations I hope to enter upon to some extent in discussing another Question proposed by me.

If, instead of (1), we had taken as our starting point the equation

$$\frac{d^2y}{dx^2} + \left[C^2\left\{(2n+1)\,x\right\}^{\frac{m-4n}{2n+1}} + \frac{n^2+n}{(2n+1)^2} \cdot \frac{1}{x^2}\right] y = 0 \text{ .... (18),}$$

we should have been led, by means of the conditions (3, 4, 5), to the transformed equation

$$\frac{d^2y}{dt^2} - \frac{2n}{t} \cdot \frac{dy}{dt} + \left\{C^2 t^{m-4n} + (n^2+n)\, t^{-2(2n+1)}\right\} t^{4n} y = 0 \text{...... (19),}$$

and thence, by means of the conditions (7, 8) and (9), to the transformation

$$\frac{d^2v}{dt^2} + C^2 t^m v = 0 \text{........................ (20),}$$

the solution of which is thus seen to depend upon that of (18). Now, in order that the process hereinbefore used may be applicable to (18), we must have

$$m - 4n = 0, \quad \text{and} \quad \frac{n^2+n}{(2n+1)^2} = -(N^2+N) \text{ .... (21, 22),}$$

the latter of which conditions is equivalent to

$$\frac{1}{4}\left\{1 - \frac{1}{(2n+1)^2}\right\} = -(N^2+N) \text{ .... (23),}$$

whence $\qquad \dfrac{1}{2n+1} = \pm(2N+1) \quad \text{or} \quad 2N+1 = \dfrac{\pm 1}{2n+1}$ .... (24, 25),

and hence, again, $\qquad N = \dfrac{1}{2}\left(\dfrac{-2n-1\pm 1}{2n+1}\right)$ .... (26),

or, in virtue of (21), $\quad N = \dfrac{1}{2}\left(\dfrac{-\frac{1}{2}m - 1 \pm 1}{\frac{1}{2}m+1}\right) = \dfrac{-m-2\pm 2}{2m+4}$........ (27),

whence $\qquad m = \dfrac{-4N-2\pm 2}{2N+1}$ .... (28).

Now, as we have seen, if N be an integer, (20), being reducible to a form solvable by indefinite integrals, is itself so solvable. Accordingly, on sub-

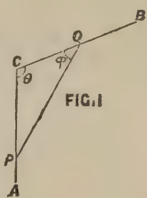
stituting for N in (28) any integers positive or negative, we are led to the ordinary forms of RICCATI, which however are here comprised in a single formula of solution. If (27) gives rise to a fractional value of N, then (20) is solvable by means of definite integrals. I have already noticed the isolated case in which the formulæ fail.

1838. (Proposed by M. W. CROFTON, B.A.)—Two steamers are continually running between a port and two given points, subtending a given angle at the port, and each of which is just visible from it; find the chance of the steamers being visible to one another at any particular instant.

Solution by the PROPOSER.

The question may be otherwise enunciated thus:—

Two equal lines CA, CB, of length r include an angle θ; to find the chance that if two points P, Q, be taken at random, one on each line, their distance PQ shall be less than r.



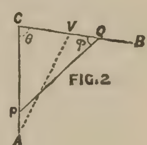
1. When $\theta > \frac{1}{2}\pi$. Let $CP = x$, $CQ = y$, $\angle CQP = \phi$; P, Q being two points such that $PQ = r$; then $x = r\sin\phi \operatorname{cosec}\theta$, $y = r\sin(\phi+\theta)\operatorname{cosec}\theta$; and if F be the measure of the favourable cases, it is easy to see that

$$F = \int_0^r x\,dy = \frac{r^2}{\sin^2\theta}\int_\theta^0 \sin\phi\cos(\phi+\theta)\,d\phi = \frac{r^2(\pi-\theta)}{2\sin\theta}.$$

Now the measure (W) of the whole number of cases is r^2; hence the required probability is

$$p = \frac{F}{W} = \frac{\pi-\theta}{2\sin\theta}.$$

2. When $\theta > \frac{1}{3}\pi$ and $< \frac{1}{2}\pi$. Draw from A (Fig. 2) a line $AV = r$; then the value of F will be



$$F = \int_{CV}^{r} x\,dy + CV\,.\,AC$$

$$= \frac{r^2}{\sin^2\theta}\int_\theta^{\pi-2\theta}\sin\phi\cos(\phi+\theta)\,d\phi + 2r^2\cos\theta;$$

whence we find $$p = \frac{3\theta-\pi}{2\sin\theta} + 2\cos\theta.$$

3. When $\theta < \frac{1}{3}\pi$, it is obvious that $p = 1$.

NOTE.—If $F(\theta)$ be the function expressing the probability in the first case, $f(\theta)$ in the second, we find they are related by the remarkable equation

$$F(\theta) + F(\pi-\theta) = f(\theta) + f(\pi-\theta).$$

From this and other examples (see solution to Quest. 1321, *Reprint*, Vol. IV., p. 86, NOTE) it would appear that the functions expressing the probabilities in problems which have different cases, are connected by very singular and simple relations, which have not yet been studied.

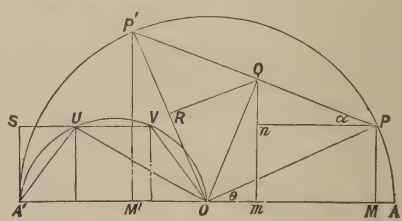
1858. (Proposed by R. BALL, M.A.)—If in any symmetric function of the differences of the roots of an equation, each root a_k be changed into $\dfrac{1}{(a_k-x)}$, show that the result, when cleared of fractions, will be a covariant.

Solution by SAMUEL ROBERTS, M.A.

It is evident that the result of making such substitution, and clearing of fractions, is a symmetric function of differences of roots and differences of x and of one or more of the roots, and each root enters term by term the same number of times. The result, therefore, is a covariant.

1883. (Proposed by R. TUCKER, M.A.)—Draw a straight line parallel to a given straight line to cut a given semicircle so that the trapezoid formed by the chord, the diameter, and the perpendiculars on the diameter from the points of section may be given or a maximum.

Solution by J. H. TAYLOR, B.A.



Let PP′ be parallel to the given line, and the angles OQP, QmA, PnQ right angles; then $\angle\, QPn = \alpha$ (a known $\angle$). Let $\angle\, POA = \theta$, and $OA = a$;

then $Pn = QP \cos\alpha$
$= a \cos\alpha \cos(\alpha+\theta)$,

and $Qm = OQ \cos\alpha$
$= a \cos\alpha \sin(\alpha+\theta)$;

hence area of trapezoid $= 2\,Pn\,.\,Qm = a^2\cos^2\alpha\sin 2(\alpha+\theta)$.

When therefore the area is constant ($= c^2$), θ can be determined from the equation

$$a^2\cos^2\alpha\sin 2(\alpha+\theta) = c^2.$$

The area is a *maximum* when $2(\alpha+\theta) = \frac{1}{2}\pi$, that is when $\theta = \frac{1}{4}\pi - \alpha$, and then OP, OP′ are at right angles to each other.

[*Otherwise:* The area of the right-angled triangle OQP′ is clearly *given* or a *maximum* at the same time with that of the trapezoid MP′; moreover the line OQ is given in position, being perpendicular to the given line to which PP′ is to be parallel; hence we readily obtain a *geometrical* solution by determining the point Q in OQ so that the perpendicular QR from Q on OP′ may be *given* or a *maximum*. This may be done by placing a line A′S perpendicular to A′O and equal to the *given* perpendicular QR, then from S drawing SUV parallel to A′O to meet a semicircle on A′O in U and V, and lastly making OQ equal to OU or OV. The triangle (and therefore the trapezoid) will be a *maximum* when U coincides with V, and then OUA′, OQP′, OQP are isosceles right-angled triangles.

The foregoing Solutions will hold good in all cases if, when PP′ cuts the diameter, the part of the trapezoid *below* AA′ be considered *negative*, in accordance with the general theory of signs.]

1888. (Proposed by E. DE JONQUIÈRES.)—(1.) Amongst the conics which have three-pointic contact with a cubic at a given point, there are, in general, three which have a three-pointic contact elsewhere and a fourth passes through the points of contact of these three with the cubic. The number of such conics is reduced to one, when the cubic has a cusp.

(2.) Amongst the conics which have four-pointic contact with a cubic at a given point there are three which touch the cubic elsewhere. There is but one such conic when the cubic has a node, and none when it has a cusp. [From the *Nouvelles Annales de Mathématiques*.]

I. *Solution by* PROFESSOR HIRST.

1. Any three distinct or coincident points ABC being taken on a cubic, and considered as fundamental points, that cubic becomes transformed, by quadric inversion, into another cubic, which likewise passes through ABC and has, in general, the same singularities as the primitive curve (see *Proceedings of the Royal Society* for March, 1865). At the same time every conic through ABC which has three pointic contact elsewhere with the primitive cubic becomes transformed into a stationary tangent of the inverse cubic. M. DE JONQUIERE'S first theorem, therefore, is the inverse of the well known one, according to which a cubic has, in general, three stationary tangents, whose points of contact are collinear.

There is but one such tangent, of course, when the cubic has a cusp.

2. The second theorem also corresponds by quadric inversion to the following very simple one:—From the point wherein a cubic is intersected by any one of its tangents three other tangents can, in general, be drawn to the curve ; only one, however, can be so drawn if the cubic has a node, and none if it has a cusp.

The theorem, however, may be demonstrated directly with equal readiness. If a, b, c, d be any four points on the cubic, it is well known that every conic through them intersects the curve again in two points which are collinear with a fixed point o on the curve, and conversely every line through o cuts the cubic in two points which lie with a, b, c, d on a conic. Now in

general four of these lines, and hence four of the conics, touch the cubic. When a, b, c, d are coincident, however, one of these four conics breaks up into two lines coincident with the tangent, and the three conics alluded to in the theorem alone remain. The modification which the theorem suffers when the cubic has a node or a cusp is manifest.

II. *Solution by* W. K. CLIFFORD.

1. Let A be the given point on the cubic, and let F be any point of inflexion, or flex. Join AF, and let AF meet the curve again in B. Then a conic may be drawn having three-pointic contact with the cubic at the points A and B. For, consider these three cubic curves :—(a) the cubic itself; (b) the line ABF taken three times over; (c) a conic having three-pointic contact at A and touching the cubic at B, together with the tangent at the flex F. The cubic (c) passes through eight out of the nine points of intersection of the cubics (a) and (b); consequently, by the theorem known as the involution of cubics, it passes through the ninth point. That is to say, a conic having three-pointic contact at A, and touching the cubic at B, will necessarily have three-pointic contact at B.

By joining the point A, therefore, to the *nine* flexes F, we shall obtain *nine* points B, and therefore nine conics fulfilling the required conditions; but only three of these points B will be real when the point A is real.

It remains to show that a conic having three-pointic contact at A passes through the three real points B. Let F_1, F_2, F_3 be the three real flexes, which are known to be in one straight line; and let B_1, B_2, B_3 be the corresponding points B. Draw a conic U having three-pointic contact at A and passing through B_1, B_2. Then consider these three cubic curves;—(a) the cubic itself; (b) the straight lines AB_1F_1, AB_2F_2, AB_3F_3; (c) the conic U and the line $F_1F_2F_3$. The cubic (c) passes through eight out of the nine intersections of the cubics (a) and (b); consequently it passes also through the ninth. That is to say, the conic U passes through the point B_3.

A cusped cubic has only one flex; in this case, therefore, the number of conics is reduced to one.

2. Let A be the given point. By COTTERILL'S Theorem (which again is a particular case of the involution of cubics), if a conic have four-pointic contact with the cubic at A, its remaining chord of intersection with the cubic will pass through a fixed point M on the curve. Now the tangent at A, taken twice over, may be regarded as a conic having four-pointic contact at A; whence it appears that the point M is the second tangential of A. The number of conics of the system which touch the cubic at some other point is therefore the number of tangents that can be drawn from M to the curve; that is, four in general, two when the cubic has a node, and one when it has a cusp. But in this number there is always included that conic which is made up of the tangent at A taken twice over; and this is not a proper solution.

THEOREM CONCERNING FIVE POINTS ON A CIRCLE.

By JOHN GRIFFITHS, M.A.

If we are given five points on the circumference of a circle of radius r (say), I propose to show that the centres of the five equilateral hyperbolas

which pass through them, taken four and four together, will lie on the circumference of another circle, whose radius is $\frac{1}{2}r$.

Let us take any two diameters of the given circle at right angles to each other as axes of coordinates; then the equation of the equilateral hyperbola which passes through the four points whose angular ordinates are $\alpha, \beta, \gamma, \delta$ can be put under the form

$$\left\{x \cos \tfrac{1}{2}(\alpha+\beta)+y \sin \tfrac{1}{2}(\alpha+\beta)-r \cos \tfrac{1}{2}(\alpha-\beta)\right\} \times$$
$$\left\{x \cos \tfrac{1}{2}(\gamma+\delta)+y \sin \tfrac{1}{2}(\gamma+\delta)-r \cos \tfrac{1}{2}(\gamma-\delta)\right\} = \lambda(x^2+y^2-r^2),$$

where $\quad \cos \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\gamma+\delta)+\sin \frac{1}{2}(\alpha+\beta) \sin \frac{1}{2}(\gamma+\delta) = 2\lambda.$

But this evidently reduces to

$$\cos \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) \,.\, (x^2-y^2) + 2 \sin \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) \,.\, xy$$
$$-2r\left\{\cos \tfrac{1}{2}(\gamma+\delta) \cos \tfrac{1}{2}(\alpha-\beta) + \cos \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma-\delta)\right\} x$$
$$-2r\left\{\sin \tfrac{1}{2}(\gamma+\delta) \cos \tfrac{1}{2}(\alpha-\beta)+\sin \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma-\delta)\right\} y+\mathrm{F}=0,$$

F being the constant term.

From this the coordinates of the centre are easily found to be

$$x = \tfrac{1}{2}r(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),\ y = \tfrac{1}{2}r(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta).$$

If, then, the angular ordinates of the five given points be $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$, the coordinates of the centres of the corresponding hyperbolas will be

$$\left.\begin{array}{l} x_1 = \frac{1}{2}r\,\Sigma \cos\phi-\frac{1}{2}r\cos\phi_5 \\ y_1 = \frac{1}{2}r\,\Sigma \sin\phi-\frac{1}{2}r\sin\phi_5 \end{array}\right\} \qquad \left.\begin{array}{l} x_2 = \frac{1}{2}r\,\Sigma \cos\phi-\frac{1}{2}r\cos\phi_1 \\ y_2 = \frac{1}{2}r\,\Sigma \sin\phi-\frac{1}{2}r\sin\phi_1 \end{array}\right\}, \text{ \&c.;}$$

whence it is easily seen that the five centres lie on the circle

$$(x-\tfrac{1}{2}r\,\Sigma\cos\phi)^2+(y-\tfrac{1}{2}r\,\Sigma\sin\phi)^2 = \tfrac{1}{4}r^2,$$

which proves the theorem.

NOTE.—Let A, B, C, D, E denote any five points on a circle; then it follows from the above that the consecutive intersections of the nine-point circles of the triangles ABC, BCD, CDE, DEA, EAB, lie on another circle, whose radius is one half that of the first.

If, however, the given points be taken on *any conic section*, the curve which passes through the five intersections in question will not in general be a circle.

ADDITION TO THE NOTE ON THE PROBLEMS IN REGARD TO A CONIC DEFINED BY FIVE CONDITIONS OF INTERSECTION. BY PROFESSOR CAYLEY.

Since writing the Note in question, I have found that a solution of Problem 7 has been given by M. De Jonquières in the paper "Du Contact des Courbes Planes, &c.," *Nouvelles Annales de Mathématiques*, Vol. III. (1864),

pp. 218—222: viz., the number of conics which touch a curve of the order n in five distinct points is stated to be

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(n^5+15n^4-55n^3-495n^2+1584n+15).$$

There are given also the following results; the number of conics which pass through two given points and touch a curve of the order n in three distinct points is

$$\frac{n(n-1)(n-2)}{2}(n^3+6n^2-19n-12),$$

and the number of conics which pass through a given point and touch a curve of the order n in four distinct points is

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(n^4+10n^3-37n^2-118n+282).$$

These formulæ are given without demonstration, and with an expression of doubt as regards their exactness—("elles sont exactes, je crois"); they apply, of course, to a curve of the order n without singularities; but assuming them to be accurate, the means exist for adapting them to the case of a curve with singularities.

[There is also a paper on the same subject in the *Annales* for January, 1866 (pp. 17—20), from the Editor's *Note* to which we have introduced a correction $(+15$ instead of $-35)$ in the formula given above.]

1876. (Proposed by R. Ball, M.A.)—If three of the roots of the equation $(a, b, c, d, e)(x, 1)^4 = 0$ be in arithmetical progression, show that

$$55296\text{H}^3\text{J}-2304a\text{H}^2\text{I}^2-16632a^2\text{HIJ}+625a^3\text{I}^3-9261a^3\text{J}^2 = 0,$$

where $\text{H} = ac-b^2$, $\text{I} = ae-4bd+3c^2$, $\text{J} = ace+2bcd-ad^2-b^2e-c^3$.

I. *Solution by* Professor Cayley.

Write $(a, b, c, d, e)(x,1)^4 = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$; then putting for a moment $\beta+\gamma+\delta = p$, $\beta\gamma+\beta\delta+\gamma\delta = q$, $\beta\gamma\delta = r$, and forming the equation

$$(\beta+\gamma-2\delta)(\beta+\delta-2\gamma)(\gamma+\delta-2\beta) = 0,$$

this is easily reduced to $\quad -2p^3+9pq-27r = 0.$

But we have $a(x^3-px^2+qx-r)(x-\alpha) = (a, b, c, d, e)(x, 1)^4$, and hence

$$p = -\frac{4b}{a}-\alpha,\ q = \frac{6c}{a}+\frac{4b}{a}\alpha+\alpha^2,\ r = -\frac{4d}{a}-\frac{6c}{a}\alpha-\frac{4b}{a}\alpha^2-\alpha^3.$$

Substituting these values of p, q, r, the foregoing equation becomes, after all reductions,

$$(20a^3,\ 20a^2b,\ -16ab^2+36a^2c,\ 128b^3-216abc+108a^2d)(\alpha, 1)^3 = 0,$$

and from this and the equation $(a, b, c, d, e)(\alpha, 1)^4 = 0$, eliminating α, we should find the condition for three roots in arithmetical progression. But it appears from the theory of invariants that the result of the elimination may be obtained by writing $b = 0$, and expressing the result so obtained in terms of a, H, I, J. Hence, writing in the two equations $b = 0$, the first equation contains the factor $4a^2$, and throwing this out, the equations become

$$5a\alpha^3 + 27c\alpha + 27d = 0,\ a\alpha^4 + 6c\alpha^2 + 4d\alpha + e = 0\,;$$

or multiplying the first by α and reducing by means of the second, the two equations become

$$5a\alpha^3 + 27c\alpha + 27d = 0,\quad 3c\alpha^2 - 7d\alpha + 5e = 0.$$

The result is of the degree 5 in the coefficients, but in order to avoid fractions in the final result it is proper to multiply it by a^4; it then becomes

$$625\, a^6e^3 - 4050\, a^5c^2e^2 + 6561\, a^4c^4e - 1890\, a^5ced^2 + 13122\, a^4c^3d^2 + 9261\, a^5d^4 = 0.$$

But writing as above $b = 0$, we have

$$a = a,\ c = \frac{\mathrm{H}}{a},\ e = \frac{\mathrm{I}}{a} - \frac{3\mathrm{H}^2}{a^3},\ d^2 = -\frac{\mathrm{J}}{a} + \frac{\mathrm{HI}}{a^2} - \frac{4\mathrm{H}^3}{a^4};$$

and substituting these values, the result is found to contain the terms $\frac{\mathrm{IH}^4}{a}$, $\frac{\mathrm{H}^6}{a^3}$ with coefficients which vanish; viz., the coefficient of the first of these terms is $+16875 + 24300 + 6561 + 7560 + 18792 - 74088, = 0$;

and the coefficient of the second of the two terms is

$$-16875 - 36450 - 19683 - 75168 + 148176,\ = 0.$$

The remaining terms give

$$\left.\begin{array}{ll} +625 & = +625\, a^3\mathrm{I}^3 \\ -5625 - 4050 - 1890 + 9261 & = -2304\, a\mathrm{H}^2\mathrm{I}^2 \\ +1890 - 18522 & = -16632\, a^2\mathrm{HIJ} \\ -18792 + 74088 & = +55296\, \mathrm{H}^3\mathrm{J} \\ +9261 & = +9261\, a^3\mathrm{J}^2 \end{array}\right\} = 0,$$

which is the required result; a more convenient form of writing it is

$$(55296\,\mathrm{J},\ -768\,\mathrm{I}^2,\ -5544\,\mathrm{IJ},\ 625\,\mathrm{I}^3 + 9261\,\mathrm{J}^2)(\mathrm{H}, a)^3 = 0.$$

Remark.—If I and J denote as above the two invariants of the form $\mathrm{U} = (a, b, c, d, e)(x, 1)^4$, and if we now use H to denote the Hessian of the form, viz.,

$$\mathrm{H} = \left\{ac - b^2,\ \tfrac{1}{2}(ad - bc),\ \tfrac{1}{6}(ae + 2bd - 3c^2),\ \tfrac{1}{2}(be - cd),\ ce - d^2\right\}(x, 1)^4,$$

then it appears by the theory of invariants that the equation of the twelfth order

$$(55296\,\mathrm{J},\ -768\,\mathrm{I}^2,\ -5544\,\mathrm{IJ},\ 625\,\mathrm{I}^3 + 9261\,\mathrm{J}^2)(\mathrm{H}, \mathrm{U})^3 = 0,$$

is such that each of its roots forms with some three of the roots of the equation $\mathrm{U} = 0$ a harmonic progression; viz., if the three roots are β, γ, δ, then we have

$$\frac{2}{x - \gamma} = \frac{1}{x - \beta} + \frac{1}{x - \delta},\ \text{or}\ x = \frac{2\beta\delta - (\beta + \delta)\gamma}{\beta + \delta - 2\gamma};$$

so that the roots of the equation of the twelfth order are the twelve values of the last mentioned function of three roots.

II. *Solution by* S. Bills.

Assume $x = y - \frac{b}{a}$; then, taking for H, I the above mentioned values, and putting $G = 2b^3 - 3abc + a^2d$, the equation in question becomes

$$y^4 + 6\frac{H}{a^2}y^2 + 4\frac{G}{a^3}y + \frac{a^2I - 3H^2}{a^4} = 0 \ldots\ldots\ldots\ldots (1),$$

Now if three of the roots of the given equation be in arithmetical progression, it is obvious that three of the roots of (1) will be so likewise, since the respective roots differ by a constant.

But the roots of (1), supposing three of them to be in arithmetical progression, are evidently of the form $p+q$, p, $p-q$, $-3p$. Taking the sum of the products of every two, of every three, and the product of all four, we shall have

$$6p^2 + q^2 = -\frac{6H}{a^2},\quad p(4p^2 - q^2) = \frac{2G}{a^3},\quad 3p^2(p^2 - q^2) = \frac{3H^2 - a^2I}{a^4} \ldots (2, 3, 4).$$

From (2), $q^2 = -\frac{6H}{a^2} - 6p^2$; substituting this in (3) and (4), we have

$$5p^3 + \frac{3H}{a^2}p - \frac{G}{a^3} = 0,\quad 21p^4 + \frac{18H}{a^2}p^2 + \frac{a^2I - 3H^2}{a^4} = 0 \ldots\ldots\ldots (5, 6).$$

But since p is a root of (1), we have

$$p^4 + 6\frac{H}{a^2}p^2 + 4\frac{G}{a^3}p + \frac{a^2I - 3H^2}{a^4} = 0 \ldots\ldots\ldots\ldots (7).$$

From (6) − 21 (7) we obtain, after a little reduction,

$$27a^2Hp^2 + 21aGp - 5(3H^2 - a^2I) = 0 \ldots\ldots\ldots\ldots (8).$$

Eliminating p between (5) and (8), as shown in my Solution of Quest. 1730 (*Reprint*, Vol. V., p. 38) and substituting for G^2 its value $-4H^3 + a^2IH - a^3J$, we obtain

$$55296\,H^3J - 2304\,aH^2I^2 - 16632\,a^2HIJ + 625\,a^3I^3 - 9261\,a^3J^2 = 0.$$

1472. (Proposed by the Editor.)—1. Find two positive rational numbers such that if from each of them, and also from the sum of their squares, their product be subtracted, the three remainders may be rational square numbers.

2. Find two positive rational numbers such that if from each of them, and also from the square root of the sum of their squares, their product be subtracted, the three remainders may be rational square numbers.

Solution by S. Bills; W. Hopps; S. Watson; *and others.*

1. Let x and y be the required numbers; then we must have

$$x - xy = \text{a square number} = p^2x^2 \text{ suppose} \ldots\ldots\ldots\ldots (1),$$
$$y - xy = \text{a square number} = q^2y^2 \text{ suppose} \ldots\ldots\ldots\ldots (2),$$
$$x^2 + y^2 - xy = \text{a square number} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2), $x = \dfrac{q^2-1}{p^2q^2-1}$, $y = \dfrac{p^2-1}{p^2q^2-1}$;

and these values being substituted in (3), it becomes (putting $\square$ to denote a rational square number)

$$(p^2-1)^2+(q^2-1)^2-(p^2-1)(q^2-1) = \square,$$

or, by putting $q+1 = p-1$,

$$(p-3)^2+(p+1)^2-(p+1)(p-3) = \square,$$

therefore $p^2-2p+13 = \square = (p-z)^2$, suppose;

whence $p = \dfrac{z^2-13}{2(z-1)}$, where z may be taken at pleasure.

If $z=2$, we have $p = -\dfrac{9}{2}$, $q = -\dfrac{13}{2}$; and two numbers satisfying the conditions are $x = \dfrac{60}{1243}$, $y = \dfrac{28}{1243}$; the three squares being

$$\left(\frac{270}{1243}\right)^2, \left(\frac{182}{1243}\right)^2, \left(\frac{52}{1243}\right)^2.$$

2. Let x and y denote the numbers, and assume $y=1-x$ and $x^2+y^2=v^2$, then the first two conditions are satisfied, and it remains to find

$$x^2+(1-x)^2 = v^2, \text{ and } v-xy=\square\ldots\ldots(4, 5).$$

From (4), $v^2-x^2 = (1-x)^2$, and to satisfy this condition assume

$$(v+x) = (p-1)(1-x), \text{ and } (p-1)(v-x) = (1+x),$$

then $x = \dfrac{p^2-2p^2}{p^2-2}$, $y = \dfrac{2p-2}{p^2-2}$, $v = \dfrac{p^2-2p+2}{p^2-2}$.

Substituting in (5), we shall have to make

$$p^4-4p^3+6p^2-4=\square=(p^2-2p+1)^2, \text{ suppose};$$

then we find $p = \frac{5}{4}$, which does not give positive values for *both* x and y; therefore assume $p = q+\frac{5}{4}$, then by substituting in the preceding expression it becomes

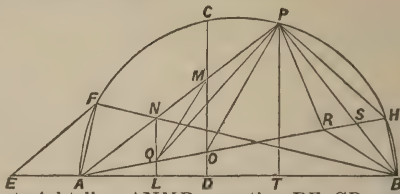
$$q^4+q^3+\frac{3}{8}q^2+\frac{65}{16}q+\frac{1}{256} = \square = \left(q^2-\frac{65}{2}q-\frac{1}{16}\right)^2, \text{ say};$$

then $q = \dfrac{4223}{264}$, and $p = \dfrac{4553}{264}$; whence we find

$$x = \frac{18325825}{20590417}, \quad y = \frac{2264592}{20590417}, \quad v = \frac{18465217}{20590417}, \quad v-xy = \left(\frac{18403967}{20590417}\right)^2.$$

1326. (Proposed by J. CONWILL.)—Find a point in the arc of a semicircle such that, if it be joined with the ends of the diameter, the quadrilateral contained between the joining lines, the diameter, and a given perpendicular to the diameter, may be a maximum.

Solution by the PROPOSER; W. HOPPS; *and others.*



Let AB be the diameter of the semicircle, and CD the given perpendicular thereto, which we will suppose to be nearer to A than to B. Produce BA to E (Euc. VI., 29) so that $AE \cdot EB = 2AD^2$; draw the chord $AF = AE$, and bisect the angle BAF by the straight line ANMP, meeting BF, CD, and the arc BC, in N, M, P respectively; then P will be the required point, and PMDB the maximum quadrilateral.

Draw NL and PT perpendicular to AB; and, H being *any other* point in the arc BC, let AH meet NL, CD, PB in Q, O, S respectively. Join PO, PH, MQ; through B draw BR parallel to PH, meeting AS in R, and join PR.

Then, because AP bisects the angle BAF, it is easy to show that the tangent at P is parallel to BF; therefore the point R will be in HQ, or in HQ produced, according as the point H is in the arc BP, or in the arc PC.

Now $AL = AF = AE$, and $2AT = BA + AF = BE$ (McDowell's *Exercises*, Prop. 77),

therefore $\quad 2AD^2 = BE \cdot AL = 2AT \cdot AL$, whence $AL : AD = AD : AT$,

therefore $\quad AQ : AO = AM : AP$;

hence PO is parallel to MQ, $\triangle PQO = PMO$, and $\triangle PQH = PMOH$.

But when H is in the arc BP, $\triangle PBH\ (=PRH) < PQH$,

therefore $\quad PMOS > \triangle SBH$, and therefore $PMDB > HODB$.

The proof applies, with a slight modification, when the point H is in the arc PC; moreover it is obvious that the required point is in the arc BC; hence the quadrilateral PMDB is a *maximum.*

[The foregoing proof (that the quadrilateral PMDB is a maximum) will apply just as well to *any segment* ACB of a circle, provided AP be drawn so as to satisfy the two following conditions:—(i.) that the tangent at P shall be parallel to BF, which will *always* be the case when AP bisects the angle BAF; (ii.) that PO shall be parallel to MQ, which will require that AD shall be a mean proportional between AT and AL, or, since $BA + AF = 2AT$,

that $\quad (BA+AF) \cdot AL = 2AD^2$ ....(A).

When ACB is a semicircle, $AL = AF$; hence, in this case, making $AE = AF = AL$, (A) becomes

$$BE \cdot EA = 2AD^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)},$$

which, together with the condition (i.), is the construction in the above solution.

When however the segment ACB is *not* a semicircle, AL is not equal to AF, and the position of AP is not given by the construction in (B).

The conditions (A) and (B) of construction may be readily obtained by the ordinary analytical processes:—thus, let $AB = a$, $AD = c$, $\angle BPA = \alpha$, $\angle BAP = \theta$; then we have

$$\text{Area of PMDB} = \tfrac{1}{2}\left\{a^2 \operatorname{cosec}\alpha \sin\theta \sin(\theta+\alpha) - c^2 \tan\theta\right\},$$

hence the quadrilateral PMDB will be a minimum when

$$c^2 \sec^2\theta = a^2 \operatorname{cosec}\alpha \sin(2\theta+\alpha), \text{ or } AM^2 = BA \cdot AF = PA \cdot AN \ldots\text{(C)},$$

that is to say, when AM and AD are mean proportionals between AP and AN, AT and AL, respectively; or, what amounts to the same thing, when

$$(BA+AF) . AL = 2AD^2, \text{ which is the condition (A).}$$

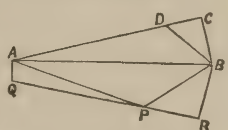
When the segment ACB is a semicircle, $\alpha = \frac{1}{2}\pi$ and (C) becomes

$a^2 \cos 2\theta \cos^2\theta = c^2$, or $(a \cos 2\theta)(a + a \cos 2\theta) = 2c^2$, or BE . EA = 2AD².]

1320. (From the *Lusus Seniles* of the Rev. JOHN SAMPSON.)—

Quid faciam, docti, carum visurus amicum,
  Quem late extensâ degere valle juvat?
Hujus ab æde domus tredecim mea millia distat,
  Quadrigis rapidis attamen ire queam
Cauponam versus distantem millia bis sex;
  Millia cauponâ quinque et amicus abest.
Quadrigis hora sex millia curritur unâ,
  Quatuor interea millia vado pedes.
Quam longe utemur quadrigis, dicite tandem,
  Tempore quo *minimo* conficiamus iter?

Solution by the Rev. W. MASON; F. COWLEY; *and others.*



The inn (C) is at a point 5 miles distant from the friend's house (B), and since AB^2 ($= 13^2 = 12^2 + 5^2$) $= AC^2 + CB^2$, the direction of the foot-path (CB) to this house from the inn is at right angles to the highway (AC).

Let D be the point to leave the highway, and put $CD = x$; then the time (in hours) along ADB is

$$t = \frac{12-x}{6} + \frac{\sqrt{(x^2+25)}}{4}; \quad \therefore -\frac{x}{6} + \frac{\sqrt{(x^2+25)}}{4} = t-2 = t',$$

and t' is obviously positive: hence putting $T = 12t'$, we have

$$9(x^2+25) = (T+2x)^2, \text{ or } 5x^2 - 4Tx + (225-T^2) = 0.$$

The *least* value of T is found from the equation

$$(4T)^2 = 4 \times 5 \times (225 - T^2), \ \therefore T = 5\sqrt{5}, \text{ and } x = \tfrac{2}{5}T = 2\sqrt{5}.$$

Hence $t = 2 + \frac{5}{12}\sqrt{5}$, and time in $AD = 2 - \frac{1}{3}\sqrt{5} = 1\frac{1}{4}$ hours, nearly.

[The following is a more general form of the problem:—

Qiven two points A, B, two numbers m, n, and any line QR, to find a point P in QR so that $\dfrac{AP}{m} + \dfrac{PB}{n}$ may be a minimum.

Draw AQ, BR perpendicular to the given line, and put $AQ = a$, $BR = b$, $QR = c$, $\angle APQ = \phi$, $\angle BPR = \psi$; then, when QR is a *straight* line, we have

$$a \cot\phi + b \cot\psi = c, \text{ and } na \operatorname{cosec}\phi + mb \operatorname{cosec}\psi = \text{minimum} \ \ldots\ldots (1, 2).$$

Differentiating (1) and (2), the condition for a minimum is found to be

$$m\cos\psi = n\cos\phi, \text{ or } \cos\phi : \cos\psi = m : n \ldots\ldots\ldots\ldots (3);$$

and the position of P is completely determined by (1) and (3).

When A and B are on opposite sides of QR, we have the case of a ray of light passing from one medium into another of different density; and the result (3) shows that the time of passing from A in one medium to B in the other is a minimum when the sines of the angles of incidence and refraction, made with a normal at P, are proportional to the velocities (m, n) of propagation of light in the two media.

When $m = n$, then by (3) the angle APQ = BPR, and we have the solution of a well-known case of the general problem; that, namely, in which AP + PB is a minimum.

In the particular case proposed in the question, QR takes the position of the highway ADC; m and n are the rates by coach and on foot, respectively; and (3) gives

$$\cos BDC = \frac{n}{m} = \frac{2}{3} = \frac{DC}{DB}; \therefore AD = \text{distance by coach} = 12 - 5\cot D = 12 - 2\sqrt{5}.]$$

1851. (Proposed by A. Renshaw.)—Given four points in a plane; show that the equation which determines the coefficient of xy, in any conic passing through the four points, so that the circumscribing rectangle may be a maximum or a minimum, is of the third order.

Solution by J. Dale; E. McCormick; *the* Proposer; *and others.*

Let $ay^2 + bxy + cx^2 + dy + ex + 1 = 0$ be the equation to a conic; then if four points in it be given, we can determine the coefficients a, c, d, e, which may therefore be regarded as known. Referring the conic to the centre as origin, and the principal axes, the above equation becomes

$$\left\{a + c \pm \sqrt{(a-c)^2 + b^2}\right\} y^2 + \left\{a + c \mp \sqrt{(a-c)^2 + b^2}\right\} x^2$$

$$= 2\left(\frac{ae^2 + cd^2 - bde}{4ac - b^2} - 1\right).$$

Hence we find that the circumscribing rectangle depends upon the function

$$\frac{ae^2 + cd^2 - bde}{(4ac - b^2)^{\frac{3}{2}}} - \frac{1}{(4ac - b^2)^{\frac{1}{2}}}.$$

Differentiating this with respect to b, and equating the result to 0, we get

$$b^3 - 2deb^2 + (3ae + 3cd^2 - 4ac)\, b - 4acd = 0,$$

and from this equation of the third degree b is to be determined.

[The question is nearly identical with the problem, To draw the least ellipse through four given points, which has been discussed by Euler in the *Petersburgh Transactions*, (Vol. IX., p. 132,) and by Messrs. Fenwick and Hearn in the *Mathematician*. (Vol. II., pp. 233, 315.)]

On the Problems in regard to a Conic defined by five Conditions of Intersection. By Professor Cayley.

There has been recently published in the *Comptes Rendus* (tom. 62, pp. 177—183, Jan. 1866) an extract of a memoir "Additions to the Theory of Conics," by M. H. G. Zeuthen (of Copenhagen). The extract gives the solutions of fourteen problems, with a brief indication of the method employed for obtaining them. Of these problems, four relate to intersections at given points, the remaining ten are included among the twenty-seven problems enumerated in my *Note* on this subject in the January Number of the *Educational Times* (*Reprint*, Vol. V., p. 25); but two of these ten are the problems 25 and 26 which are in my *Note* stated to have been solved; there are, consequently, of the twenty-seven problems, in all twelve which

No. of Prob.	1, 8, 10, 12, 14, 17, 19, 21, 23, 25, 26, 27
Zeuthen's No.	—, 14, 13, 11, 8, 3, 12, 7, 2, 6, 1, —

are solved: viz., these are where it is to be observed that Zeuthen's solutions apply to the case of a curve of a given order with given numbers of double points and cusps. The problems 25 and 26 had been previously solved only in the case of a curve without singularities. As to Prob. 27, the solution mentioned in my former *Note* is in fact applicable to the general case. The solution for Prob. 1 may also be extended to this general case, viz., for a curve of the order m with δ double points and κ cusps the required number is $= m(12m-27)-24\delta-27\kappa$; or, if n be the class, then this number is $= 12n-15m+9\kappa$: so that all the twelve problems are solved in the general case.

The results obtained by M. de Jonquières, as stated in my *Note* in the March Number (*Reprint*, Vol. V., p. 57), seem to be all of them erroneous. In fact, for the number of conics passing through two given points and touching a curve of the order m in three distinct points (which is a particular case of Prob. 23), Zeuthen's formula applied to a curve without singularities gives this

$$= \tfrac{1}{6}m\,(m-2)\,(m^4+5m^3-17m^2-49m+108)$$

instead of the value $\tfrac{1}{2}m\,(m-1)\,(m-2)\,(m^3+6m^2-19m-12)$ which is

$$= \tfrac{1}{6}m\,(m-2)\,(m^4+5m^3-25m^2+7m+12);$$

and I have by my own investigation verified Zeuthen's Number. So for the number of conics through a given point and touching a curve of the order m in four distinct points (which is a particular case of Prob. 17), Zeuthen's formula applied to a curve without singularities gives this

$$= \tfrac{1}{24}m\,(m-2)\,(m-3)\,(m^5+9m^4-15m^3-225m^2+140m+1050)$$

instead of the value

$$\tfrac{1}{24}\,m\,(m-1)\,(m-2)\,(m-3)\,(m^4+10m^3-37m^2-118m+282)$$

which is

$$= \tfrac{1}{24}\,m\,(m-2)\,(m-3)\,(m^5+9m^4-47m^3-81m^2+400m-282),$$

and it may I think be inferred that the expression obtained for the number of conics which touch a given curve in five distinct points (Prob. 7), containing as it does the factor $(m-1)$, is also erroneous.

I have obtained for Prob. 2 a solution which I believe to be accurate; viz., the number of the conics (4, 1), (that is, the conics which have with a given curve a 5-pointic intersection and also a 2-pointic intersection, or ordinary contact), is

$$=10n^2+10nm-20m^2-130n+140m+10\kappa(m+n-9)-4[(n-3)\kappa+(m-3)\iota]$$

where ι (the number of inflexions) is $=3n-3m+\kappa$, but I prefer to retain the foregoing form, without effecting the substitution.

1495. (Proposed by HUGH GODFRAY, M.A.)—Show that $\frac{1}{3}n\,(n-1)\,(n-2)$ points can always be so arranged in a plane that they shall be situated by eights in $\frac{1}{24}n\,(n-1)\,(n-2)\,(n-3)$ circles.

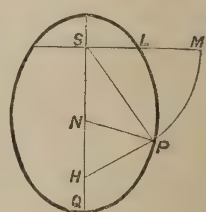
Solution by SAMUEL ROBERTS, M.A.

If n points be taken four together, we shall have $\frac{1}{24}n\,(n-1)\,(n-2)\,(n-3)$ sets. Each set, considered as a quadrangle, determines a circle passing through the intersections of the diagonals and the opposite sides. To each triangle of the quadrangle, correspond $(n-3)$ such circles. The number of triangles being $\frac{1}{6}n\,(n-1)\,(n-2)$, the whole series of circles can be formed into the same number of sets of $(n-3)$. Since four triangles belong to a quadrangle, each circle will reappear four times in the sets; and two circles will not occur together more than once. All this is of course on the assumption that there is no special limitation of the points, and all the circles are different.

Suppose that the $(n-3)$ circles of each set pass through the same two points. This supposition is permissible, since no two circles occur together twice. It follows that all the points of intersection in question are of the $(n-3)$rd order, and each circle has upon it 8 such points, while the total number of them is $\frac{1}{3}n\,(n-1)\,(n-2)$.

1641. (Proposed by E. McCORMICK.)—An ellipse is placed with its major axis vertical; find, geometrically, the straight line of quickest descent from the upper focus to the curve.

Solution by G. C. DE MORGAN, M.A.; *and the* REV. J. L. KITCHIN, M.A.



Let S, H be the foci. Draw SLM perpendicular to the major axis, making LM equal to the semi-parameter SL. Describe a circle with centre S and radius SM, meeting the ellipse in P; then SP is the straight line required.

For, making the angle NPS = NSP, we have, if e be the eccentricity of the ellipse,

$$\cos \mathrm{NSP} = \frac{1}{e}\left(1 - \frac{\mathrm{SL}}{\mathrm{SP}}\right) = \frac{1}{2e},$$

whence it follows that $\mathrm{SN} = \mathrm{NP} = e\,.\,\mathrm{SP}$. Hence

since SH $= e$(SP + HP), we have SN : NH = SP : PH, therefore NP is the normal at P, and the proposition follows at once. [For a circle drawn round N as centre with radius NP would pass through S and *touch* the ellipse at P.]

If $e < \frac{1}{2}$, the circle never meets the ellipse, but in that case, Q being the lower end of the major axis, the circle on SQ as diameter lies wholly within the ellipse, and SQ is the required line.

1798. (Proposed by Professor SYLVESTER.)—

(1.) Let $f(x) = 1 + x + \dfrac{x^2}{1\,.\,2} + \dfrac{x^3}{1\,.\,2\,.\,3} \dots + \dfrac{x^n}{1\,.\,2 \dots n}$,

prove that $f(x)$ cannot have two real roots.

(2.) Let $\phi(x) = 1 + \nu x + \dfrac{\nu(\nu+1)}{1\,.\,2} x^2 + \dots + \dfrac{(\nu+1)\,..\,(\nu+n-1)}{1\,.\,2\,..\,n} x^n$,

if $\nu > 0$ or $< -n$, prove that $\phi(x)$ cannot have two real roots.

(3.) Deduce (1) from (2).

I. *Solution by the* PROPOSER.

If $\mathrm{F}(x) = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n$, and we write

$$a_0 = c_0, \quad a_1 = \frac{c_1}{\mu}, \quad a_2 = \frac{1\,.\,2\,.\,c_2}{\mu(\mu+1)}, \quad \dots\dots \quad a_n = \frac{1\,.\,2\,.\,3\,..\,n}{\mu(\mu+1)\,..\,(\mu+n-1)}\,.\,c_n,$$

it has been shown (see the late Mr. PURKISS' paper in the *Messenger of Mathematics*, Vol. III., pp. 129—142) that the number of imaginary roots in Fx cannot fall short of the number of changes of sign in the progression

$$a_0^2, \quad a_1^2 - a_0 a_2, \quad a_2^2 - a_1 a_3, \quad \dots\dots \quad a_{n-1}^2 - a_{n-2} a_n, \quad a_n^2,$$

provided that μ is not intermediate between 0 and $-n$.

In (2), changing x into $\dfrac{1}{x}$, we have $x^n + \nu x^{n-1} + \dfrac{\nu(\nu+1)}{1\,.\,2} x^{n-2} + \,..\, = 0$. Hence, making $\mu = \nu$, we have $a_0 = 1$, $a_1 = 1$, ... $a_n = 1$; and the criterion series becomes 1, 0, 0, ... 0, 1; which, since any zero is free to be taken + or − indifferently, is equivalent to the series $+ - + - \dots \pm +$, whence it is evident that all, or all but one, of the roots are imaginary.

We may derive (1) from (2) by taking $\nu = \infty$ and writing x for νx, or at once from the general theorem by taking $\mu = \infty$.

If we wished to apply Newton's rule (as given by Newton) to the equation $fx = 0$, *i. e.*, $x^n + n x^{n-1} + n(n-1) x^{n-2} + \dots = 0$, his criterion series, omitting square factors, being what my series becomes when μ is taken equal to n, would be

$$1; \quad 1^2 - 2; \quad 2^2 - 1\,.\,3; \quad 3^2 - 2\,.\,4; \quad \dots\dots \quad (n-1)^2 - n(n-2); \quad 1;$$

$$i.e. \; 1; \quad -1; \quad 1; \quad 1; \quad \qquad 1; \quad 1;$$

and as there are but *two* changes of sign in the above, we could infer the certain existence of not more than *one pair* of imaginary roots, which well illustrates the importance of the arbitrary parameter (limited) which I have imported into the theorem.

II. *Solution by the* REV. J. BLISSARD.

This Question proposed in its most general form would be as follows:—

Let $F_n x = \dfrac{a_0 x^n}{1.2\ldots n} + \dfrac{a_1 x^{n-1}}{1.2\ldots(n-1)} + \ldots + a_{n-1}x + a_n.$

Required the conditions necessary to be fulfilled in order that $F^n x = 0$ must (n even) have all its roots unreal, or (n odd) all unreal but one.

Let $F_1 x = a_0 x + a_1$ (a_0 positive), $F_2 x = \dfrac{a_0 x^2}{1.2} + a_1 x + a_2$, &c.

It is evident that $F_2 x$, $F_3 x$ .. $F_n x$ are all obtained by successive Integration from $F_1 x$, the constants $a_2, a_3 \ldots a_n$ being respectively added. Also, since $F_{n-1}x$ is derived from $F_n x$ by Differentiation, $F_n x = 0$ must have at least as many unreal roots as $F_{n-1}x = 0$. Hence if n is odd and all the roots of $F_{n-1}x = 0$ are unreal, $F_n x = 0$ can only have one real root; and if n is even and $F_{n-1}x = 0$ has only one real root, $F_n x = 0$ must have at least $n-2$ unreal roots. It is now required to determine the conditions under which $F_n x = 0$ (n even) *must* have all its roots unreal. For this purpose it is evidently necessary that a_n should be positive and > 0, since otherwise two roots of $F_n x = 0$ must be real. Now let a_n receive all values from 0 to $+\infty$; then $F_n x = 0$ must begin ($a_n = 0$) with having two real roots (one being zero) and end ($a_n = \infty$) with having all its roots unreal, for in this case the equation $F_n x = 0$ is reduced to $x^n = -\infty$, all the roots of which must be unreal. Hence, as a_n passes from 0 to ∞, two roots of $F_n x = 0$ must pass from reality through equality to unreality. There must consequently be a positive value of a_n which renders two roots of $F_n x = 0$ real and equal, and $F_{n-1}x = 0$ must have one root the same as these. It follows that if x be eliminated between $F_n x = 0$ and $F_{n-1}x = 0$, the result of this elimination (denoted by C_n) when arranged according to the powers of a_n must give the conditions of reality, equality, and unreality of two roots of $F_n x = 0$, according as that result, viz. C_n, is negative, zero, or positive. Hence, if C_2, C_4, &c. be the results respectively of the elimination of x between $F_1 x = 0$ and $F_2 x = 0$, between $F_3 x = 0$ and $F_4 x = 0$, and so on, these functions, viz. C_2, C_4, &c. will be $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ in number, as n is even or odd; and if they are all positive, then $F_n x = 0$ must (n even) have all its roots unreal, and (n odd) all unreal but one.

If r is the real root in $F_{n-1}x = 0$, then $F_{n-1}r = 0$; and if $F_n r$ is positive, all the roots of $F_n x = 0$ (n even) must be unreal. For if a_n be of such a value that two roots of $F_n x = 0$ are equal, r must be that root, and $F_n r = 0$. Hence, as a_n varies from 0 to ∞, $F_n r$ will be negative, zero, or positive, according as $F_n x = 0$ has two roots which pass from reality, through equality, to unreality, *i.e.*, if $F_n r$ is positive, all the roots of $F_n x = 0$ are unreal.

The conditions therefore necessary to be fulfilled in order that $F_n x = 0$ may be incapable of having more than one real root may be thus stated.

1. The coefficients a_0, a_2, a_4, &c. must all be positive.

2. C_2, C_4, C_6, &c. must all be positive.

3. Or if the real root in $F_1 x = 0$, $F_3 x = 0$, &c. be respectively r_1, r_3, &c., then $F_2 r_1$, $F_4 r_3$, &c. must all be positive.

The second and third of these sets of conditions must of course involve the same principle, so that if one holds good the other must hold good also. We can however always obtain the second set of conditions by mere elimination, but the third set, except in particular cases, will require the solution of equations.

Professor SYLVESTER'S Question may now be solved by proving that if, for any positive integral value of n, $\phi x=0$ is incapable of having more than one real root, this must also hold good for the next value of n, and therefore generally, provided ν is positive or $< -n$.

First, let n be odd, and let $\phi' x = \dfrac{d\phi x}{dx}$; then, by hypothesis, $\phi' x=0$ can have no real root, and therefore $\phi x=0$ can only have one real root.

Next, let n be even, then, by hypothesis, $\phi' x=0$ has only one real root. Let that root be α; then we have

$$0 = 1+(\nu+1)\alpha + \frac{(\nu+1)(\nu+2)}{1\,.\,2}\alpha^2 + \,..\, + \frac{(\nu+1)\,..\,(\nu+n-1)}{1\,.\,2\,..\,(n-1)}\alpha^{n-1}, \text{ and}$$

$$\phi\alpha = 1+\nu\alpha + \frac{\nu(\nu+1)}{1\,.\,2}\alpha^2 + \,..\, + \frac{\nu(\nu+1)\,..\,(\nu+n-1)}{1\,.\,2\,..\,n}\alpha^n,$$

therefore, by subtracting, we obtain

$$\phi\alpha = -\left(\alpha+(\nu+1)\alpha^2 + \frac{(\nu+1)(\nu+2)}{1\,.\,2}\alpha^3 + \,..\, + \frac{(\nu+1)\,..\,(\nu+n-2)}{1\,.\,2\,..\,(n-2)}\alpha^{n-1}\right) + \frac{\nu(\nu+1)\,..\,(\nu+n-1)}{1\,.\,2\,..\,n}\alpha^n$$

$$= \left\{\frac{(\nu+1)\,..\,(\nu+n-1)}{1\,.\,2\,....\,(n-1)} + \frac{\nu(\nu+1)\,..\,(\nu+n-1)}{1\,.\,2\,..\,n}\right\}\alpha^n = \frac{(\nu+1)(\nu+2)\,..\,(\nu+n)}{1\,.\,2\,..\,n}\alpha^n$$

which, since n is even, is necessarily positive, provided ν be positive or $< -n$.

But it has been shown that if $\phi' x=0$ has but one real root, and if that root, substituted in ϕx, gives a positive result, the roots of $\phi x=0$ must be all unreal; hence when n is even, $\phi x=0$ must have all its roots unreal. If therefore the hypothesis holds good for $n=2$, which is easily shown to be the case, it must hold good for $n=3$ and $n=4$, &c., and therefore for $n=5$ and $n=6$, and so on generally.

Again, let ν be indefinitely great, then ϕx is reduced to

$$1+\nu x + \frac{\nu^2 x^2}{1\,.\,2} + \cdots + \frac{\nu^n x^n}{1.2..n},$$

and if x be assumed indefinitely small, νx may be assumed to equal any finite quantity. For νx put x, and for ϕx put fx; then $fx=0$ must be incapable of having more than one real root.

1817. (Proposed by M. Collins, B.A.) — In lines of the third order, prove that the locus of middle points of chords parallel to an asymptote which does not *cut the curve,* is a straight line; but when the asymptote cuts the curve, show that the locus then becomes a hyperbola.

Solution by Professor Cremona.

Ce théorème n'est qu'une corollaire très-simple d'une propriété connue (*Introd. ad una Teoria geom. delle curve piane*, 139). Les cordes d'une cubique plane parallèles à une asymptote ont leurs milieux dans une conique (la conique polaire du point à l'infini sur l'asymptote), qui a cette asymptote en commun avec la cubique, et est par conséquent une hyperbole. Cela arrive toujours si le point à l'infini sur l'asymptote est un point ordinaire de la cubique. Mais si ce point est un point d'inflexion (ce qui revient à supposer que l'asymptote ne rencontre pas la cubique à distance finie), la conique polaire se décompose en deux droites, dont l'une est l'asymptote même, et l'autre, lieu effectif des points-milieux des cordes, est nommée *polaire harmonique* du point d'inflexion.

1835. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Three lines being drawn at random on a plane, determine the probability that they will form an acute triangle.

Solution by the Proposer.

The condition in the question, abstractedly considered, depends on the circumstances of direction only without any regard to position or magnitude. And every possible direction will be comprised in an angular range of two right angles, since lines drawn in opposite directions may here be properly treated as identical, the idea being, in fact, that of linear direction apart from any consideration of an origin or zero point of linear magnitude. For simplicity in estimating the angles, let a right angle be taken as the unit of measurement; and suppose two lines, drawn at random, and prolonged indefinitely, to intersect each other at supplemental angles x, $2-x$, where x may be any positive value less than unity. Then in order that the triangle completed by drawing a third line may be acute, the vertical angle must evidently be the acute angle x; and the sum of the second and third angles of the triangle must therefore be equal to $2-x$. Moreover the second and third angles must also be both of them acute; and therefore it follows that each of them must exceed $1-x$ and not exceed 1. The direction of the third line is thus restricted to an angular range of x, within which the three angles of the resulting triangle are severally acute. Therefore, as the entire unlimited range in the direction of this line is 2, and as the elements comprised in those ranges are obviously equally admissible, the probability of an acute triangle, under the hypothesis of the assumed lines, is equal to $\frac{1}{2}x$. Hence, as the two lines originally assumed may indifferently have every value of x from 0 to 1, the required absolute probability of an acute triangle is obtained by multiplying by dx and integrating between those limits, and is $\frac{1}{4}$.

NOTE.—The probability involved in this question is the same as that in which the three angular points are taken at random in the circumference of a given circle; as is evident from the consideration that, when an angular point is supposed to pass over equal arcs of the circle, the directions of the adjacent sides will describe equal angles.

II. *Solution by* J. M. WILSON, M.A.; E. MCCORMICK; M. COLLINS, B.A.; E. FITZGERALD; *and others.*

The probability of the acute angle made by two lines lying between θ and $\theta + d\theta$ is $\frac{2d\theta}{\pi}$, and all lines whose directions lie between the perpendiculars to the two lines will make acute triangles with them; hence in this case the probability will be $\frac{\theta}{\pi}$. Therefore the probability required will be

$$p = \int_0^{\frac{1}{2}\pi} \frac{\theta}{\pi} \cdot \frac{2d\theta}{\pi} = \frac{1}{4}.$$

1873. (Proposed by C. W. MERRIFIELD, F.R.S.)—Assuming that all lives are of equal duration, what must that duration be, in order that the births, deaths, and consequent increase or decrease of population, may remain unchanged?

Solution by the PROPOSER; *and* J. H. TAYLOR, B.A.

Let the ratio of births to a unit of population be b, and that of the deaths d, then the rate of increase is $r = b - d$; and let t be the length of the period over which the b and d are reckoned.

If b were the same as d, or $r = 0$, we should have simply $l = \frac{t}{d}$, l being the duration of life.

If we consider the increase in the period t to be r, then the momentary rate of increase will be $\log_\epsilon (1+r)$; for, if we consider the instantaneous rate of increase to be ρ, we have $\epsilon^\rho = 1 + r$. We must therefore affect d with the factor $\frac{\log(1+r)}{r}$ to bring it to the momentary standard; for the equation $r = b - d$ is true for any period, and at the limit also.

Hence we have $l = \dfrac{rt}{d \log_\epsilon (1+r)}$.

1877. (Proposed by J. GRIFFITHS, M.A.)—Let P be the intersection of the three perpendiculars; O the centre of the circumscribed circle (radius

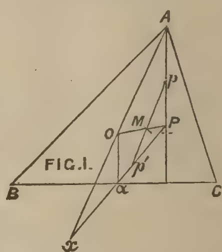
$=\text{R}$); α, β, γ the middle points of the sides, of any triangle ABC. On the segments PA, PB, PC let the three points p, q, r be taken, such that $\text{P}p=\frac{1}{n}\,.\,\text{PA}$, $\text{P}q=\frac{1}{n}\,.\,\text{PB}$, $\text{PR}=\frac{1}{n}\,.\,\text{PC}$; and on $\text{P}\alpha$, $\text{P}\beta$, $\text{P}\gamma$ three other points p', q', r', such that $\text{P}p'=\frac{2}{n}\,.\,\text{P}\alpha$, $\text{P}q'=\frac{2}{n}\,.\,\text{P}\beta$, $\text{P}r'=\frac{2}{n}\,.\,\text{P}\gamma$. Prove (1) that the lines pp', qq', rr' intersect on the line PO in a point M, such that $\text{PM}=\frac{1}{n}\,.\,\text{PO}$; (2) that the six points in question lie on a circle whose centre coincides with M, and whose radius $=\frac{1}{n}\,.\,\text{R}$; (3) that this circle will touch the circle inscribed in the triangle, if $\frac{1}{n}=\frac{1}{2}$ or $=1+\frac{r^2}{\rho^2}$, where r, ρ are the radii of the inscribed and self-conjugate circles of the triangle.

Solution by the Proposer; J. H. Taylor, B.A.; *and others.*

1. Produce AO, $\text{P}\alpha$ (Fig. 1) to meet in x; and join $\text{O}\alpha$; then, by similar triangles, we have

$$x\text{O} : x\text{A} = \alpha\text{O} : \text{PA} = 1 : 2,$$

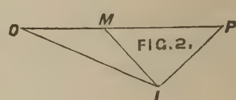
therefore O is the point of bisection of $\text{A}x$, and α that of $\text{P}x$. Again, since $\text{P}p : \text{PA} = \text{P}p' : \text{P}x$, the line pp' is parallel to $\text{A}x$, and must therefore cut PO in a point M such that $\text{PM}=\frac{1}{n}\,.\,\text{PO}$ and $p\text{M}=p'\text{M}=\frac{1}{n}\,.\,\text{AO}$. In the same way qq' and rr' may be shown to intersect the line PO in M.



2. Hence, since AO = BO = CO = R, the six points p, p'; q, q'; r, r' will lie on a circle whose centre is M and radius $\frac{1}{n}$ R.

3. Let I (Fig. 2) be the centre of the inscribed circle of the given triangle; then, if the circles (M) and (I) touch each other, we must have

$$\text{MI}=\frac{1}{n}\,.\,\text{R}\pm r.$$



But $\dfrac{\text{PO}^2+\text{PI}^2-\text{OI}^2}{2\,\text{PO}\,.\,\text{PI}}=\cos\text{OPI}=\dfrac{\text{PM}^2+\text{PI}^2-\text{MI}^2}{2\,\text{PM}\,.\,\text{PI}}$, and $\text{PM}=\frac{1}{n}\,.\,\text{PO}$;

therefore $\text{PO}^2+\text{PI}^2-\text{OI}^2=n\,(\text{PM}^2+\text{PI}^2-\text{MI}^2)$; also $\text{OI}^2=\text{R}^2-2\text{R}r$,

moreover $\qquad \text{PI}^2=2r^2-4\text{R}^2\cos\text{A}\cos\text{B}\cos\text{C}=2r^2+\rho^2,$

and $\qquad \text{PO}^2=\text{R}^2-8\text{R}^2\cos\text{A}\cos\text{B}\cos\text{C}=\text{R}^2+2\rho^2;$

therefore $\qquad (2r^2+3\rho^2+2\text{R}r)=n\left\{\dfrac{\text{R}^2+2\rho^2}{n^2}+2r^2+\rho^2-\left(\dfrac{\text{R}}{n}\pm r\right)^2\right\},$

or $\qquad (2r^2+3\rho^2+2\text{R}r)\,n=(r^2+\rho^2)\,n^2\mp2\text{R}rn+2\rho^2.$

If we take the + sign, we shall have $(r^2+\rho^2)n^2+2\rho^2=(2r^2+3\rho^2)n$,

whence we obtain $$\frac{1}{n}=1+\frac{r^2}{\rho^2}, \text{ or } \frac{1}{2}.$$

The other values of n for which the circles touch each other are evidently given by the equation $(r^2+\rho^2)n^2+2\rho^2=(2r^2+3\rho^2+4Rr)n$.

When $n=\frac{1}{2}$, the circle becomes the well-known nine-point circle.

1879. (Proposed by T. Cotterill, M.A.)—If forces represented by the sides of a plane hexagon taken in order are in equilibrium, the directions of the sides of the two triangles formed by joining alternate points of the hexagon are in involution.

Solution by the Rev. R. Townsend, M.A.

Denoting the three pairs of opposite vertices of the hexagon by A and A′, B and B′, C and C′; and replacing the three pairs of forces AB′ and AC′, BC′ and BA′, CA′ and CB′, acting at the three vertices A, B, C of either triangle, by their three resultants AX, BY, CZ, which are evidently parallel to the three corresponding sides B′C′, C′A′, A′B′ of the other; then since the three lines AX, BY, CZ, as representing by hypothesis three forces in equilibrium, are concurrent, therefo e the theorem follows; every three concurrent lines through the three vertices of any triangle determining with the three parallels through the point of concurrence to the three sides of the triangle a system of six rays in involution. (Townsend's *Modern Geometry*, Art. 368, Ex. 2.)

Since, when a number of forces in a plane are represented in magnitude and direction by the several sides of a polygon in the plane, the sum of their moments round any point in the plane is represented by double the area of the polygon (*Modern Geometry*, Art. 118); it follows at once, as an immediate corollary from the above, that *when the area of a plane hexagon* $=0$, *the directions of the sides of the two triangles determined by its two triads of alternate vertices are in involution.*

1909. (Proposed by the Rev. R. H. Wright, M.A.)—If λ and λ' be the angles which any two conjugate diameters AB and CD of an ellipse subtend at any point P in the curve, and α the angle which either axis subtends at an extremity of the other axis; prove that $\cot^2\lambda+\cot^2\lambda'=\cot^2\alpha$.

Solution by S. W. Bromfield; J. H. Taylor, B.A.; *the* Proposer; *and others.*

Let the points P, A, C be respectively denoted by $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$, $(a\cos\phi', b\sin\phi')$, where $\phi'=90^\circ+\phi$; then the tangents

of the angles which AP and BP make with the major axis are $\frac{b}{a} \cdot \frac{\sin\phi \mp \sin\theta}{\cos\phi \mp \cos\theta}$;

therefore $\tan\lambda = \frac{2ab}{a^2-b^2} \cdot \frac{\sin(\theta-\phi)}{\cos^2\phi - \cos^2\theta} = \frac{2ab}{a^2-b^2} \cdot \frac{1}{\sin(\theta+\phi)}$;

similarly $\tan\lambda' = \frac{2ab}{a^2-b^2} \cdot \frac{1}{\sin(\theta+\phi')} = \frac{2ab}{a^2-b^2} \cdot \frac{1}{\cos(\theta+\phi)}$,

therefore $$\cot^2\lambda + \cot^2\lambda' = \left(\frac{a^2-b^2}{2ab}\right)^2 = \cot^2\alpha.$$

Angular and Linear Notation.

A Common Basis for the *Bilinear* (a transformation of the *Cartesian*), the *Trilinear*, the *Quadrilinear*, &c., Systems of Geometry. By H. McColl.

Def. 1. Let any two straight lines X and Y intersect in a point S. From S as centre and with any radius describe a circle cutting X in the point A, and Y in the point B. The angle ASB is proportional to the arc BA, measured in the *positive* (or *unscrewing*) direction of the circumference *from the point* B *to the point* A; and the angle BSA is proportional to the remaining portion of the circumference, measured in the *same* (positive) direction *from the point* A *to the point* B. Hence, if θ be the circular measure of ASB, $2\pi-\theta$ will be the circular measure of BSA.

Def. 2. The *Point of Reference* (denoted by P) is a point taken in a plane indicating the directions in which angles and distances are to be measured. It must be understood *that no line in its plane, mentioned in the course of an investigation, passes through it, and that it is situated on the positive side of every line in the investigation.* In other respects its position is arbitrary.

Def. 3. Let X, Y, Z, U, V, &c., be any straight lines in the same plane. Let p be the general expression for every point in some locus W. Then x, y, z, u, v, &c., denote respectively *the lengths of the perpendiculars from* p *upon the straight lines* X, Y, Z, U, V, &c. Any one of these distances, say x, is understood to be *positive* when p and P are *on the same side* of the corresponding line X, and *negative* when *on opposite sides of* X. Hence, the values of x, y, z, u, v, &c., vary with the position of p. The perpendiculars x, y, z, &c., are called the *coordinates* of the point p.

Def. 4. The symbols x y and x·y, are called respectively the first and second inclinations of X to Y. Interpretation:—Let X and Y intersect in S. From S as centre, and with any radius, describe a circle cutting X *on the positive side* of Y in the point x_1 and *on the negative side* of Y in the point x_2, and also cutting Y *on the positive side* of X in the point y. Then x y denotes the angle x_1 s y, and x·y denotes the angle x_2 s y; these angles being interpreted as in Def. 1. Hence y x $=2\pi-$ x y and x·y $=$ x y $\pm\pi$, the positive or negative sign to be taken according as x y is less or greater than π. It will be observed that the sine, the cosine, the tangent, &c., of x·y are not affected by the double sign.

Def. 5. When every point in a locus W is subject to any condition $f(x, y, z, u, v, \&c.) = 0$, this is denoted by $(w) = f(x, y, z, u, v, \&c.)$, which must be considered as a brief expression for the sentence,—"The equation to the locus W is $f(x, y, z, u, v, \&c.) = 0$." The symbol (w) may be called the *zero expression of the line* (or locus) W. When the zero expression contains *two* variable coordinates, it is called *bilinear;* when it contains *three*, *trilinear*, and so on. Similarly, an investigation *considered as a whole* may be called *bilinear, trilinear, quadrilinear*, &c., according to the number of variable coordinates employed throughout it. Every *zero expression*, for instance, may be *bilinear*, while the whole investigation is *quintilinear.*

The following are a few applications of this notation.

1. Let X, Y, Z be any three straight lines in the same plane. Whatever be the position of the point of reference P, any one of the three, say Z, is connected with the other two by the relation

$$(z) = x \sin \mathrm{y\,z} + y \sin \mathrm{z\,x} - z' \sin \mathrm{x\,y},$$

in which z' denotes the perpendicular upon Z from the intersection of X and Y.

When either of the coefficients sin y z, sin z x, becomes zero, the other becomes numerically equal to sin x y; so that we shall have

$$(z) = y \pm z'; \quad (z) = x \pm z';$$

the signs depending upon the position of P. The first is the equation to Z when parallel to Y; the second is the equation to Z when parallel to X. When $z' = 0$, we shall have

$$(z) = x \sin \mathrm{y\,z} + y \sin \mathrm{z\,x},$$

which is the equation to Z when passing through the intersection of X and Y. When, in the last equation, the coordinates x and y are of the same sign *for the same position of p*, we shall see that sin y z and sin z x have opposite signs; and when x and y have opposite signs, sin y z and sin z x have the same sign. When sin y z and sin z x become *numerically* equal, the last equation becomes

$$(z) = x \pm y.$$

An examination of Defs. 3 and 4 will make it clear that when the *positive* sign is taken Z bisects the angle x·y, and that when the *negative* sign is taken Z bisects x y.

By means of this equation we can immediately show that, *whatever be the position of* P, the bisectors of the three angles x y, y z, z x *meet in a point.* For if U, V, W be respectively the three bisectors, we shall have

$$(u) = x - y; \quad (v) = y - z; \quad (w) = z - x;$$

and the values of x and y which satisfy (u) and (v) simultaneously will also satisfy (w). The point at which the three bisectors meet will be found within or without the triangle intercepted by the lines X, Y, Z, according as the point of reference P is within or without this triangle. The solution of this problem, therefore, on the principle of *Angular and Linear Notation*, is more comprehensive than that usually given.

2. Let X, Y, Z be the sides of any triangle, understood to have the same signs respectively as their proportionals sin y z, sin z x, sin x y; and let

z' be the perpendicular upon Z from the intersection x y. Whatever be the position of P we shall have

$$(z) = \mathrm{X}x + \mathrm{Y}y - \mathrm{Z}z',$$

in which it will be observed that $\mathrm{Z}z' =$ twice area of triangle. If P be taken within the triangle, and U, V, W be respectively the bisectors of Z, X, Y from the opposite angles, we shall have

$$(u) = \mathrm{X}x - \mathrm{Y}y\,; \quad (v) = \mathrm{Y}y - \mathrm{Z}z\,; \quad (w) = \mathrm{Z}z - \mathrm{X}x,$$

which shows that U, V, and W meet in a point.

Let X, Y, Z, U be any straight lines in the same plane, we shall have, as in the usual trilinear system,

$$(u) = lx + my + nz,$$

in which l, m, n are constant quantities.

In Cartesian language, let x, y be the coordinates of any point p referred to oblique axes. For "axis of y" read "line X;" for "axis of x" read "line Y;" interpret x and y as in Def. 3, that is, as the perpendiculars from p upon X and Y respectively: then, if a and b be the values of x and y in the Cartesian language, and a_1 and b_1 be the values of x and y when interpreted as in Def. 3, we shall have $a_1 = a \sin \mathrm{x\,y}$, and $b_1 = b \sin \mathrm{x\,y}$.

Hence, any equation expressed in Cartesian language may immediately be transformed into another expressed in bilinear (perpendicular) coordinates. For example, the equation to the circle referred to oblique axes is

$$(x-a)^2 + (y-b)^2 + 2\,(x-a)\,(y-b) \cos \omega - c^2 = 0,$$

in which a, b are the coordinates of the centre, c denotes the radius, and ω the inclination of the axes. The same equation expressed in bilinear (perpendicular) coordinates is

$$(z) = (x-a_1)^2 + (y-b_1)^2 + 2\,(x-a_1)\,(y-b_1) \cos \mathrm{x\,y} - {c_1}^2,$$

in which Z denotes the circumference, x and y are the perpendiculars from *any* point in Z upon the lines X and Y respectively, a_1 and b_1 are the perpendiculars from the centre upon X and Y respectively, and c_1 is the radius multiplied by sin x y.

The advantages of the Notation here proposed may be summed up as follows :—

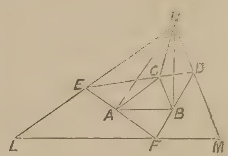
By substituting the more comprehensive notion of a *Point of Reference* for that of a fixed *Origin*, adopting *perpendicular* coordinates (denoted by Italic letters) in all cases, and removing all the unnecessary restrictions imposed by the conventions of fixed *axes*, *triangles*, and *quadrilaterals* of reference, the *Linear* Notation will enable us to unite the *Cartesian* and modern *Trilinear*, &c. methods in one harmonious system of Analytical Geometry.

The *Angular* Notation, on the other hand, supplies us with brief and *distinct* expressions for *all* the various angles resulting from the intersections of any number of straight lines in the same plane. The symbols x y. y x, x·y and y·x are four distinct expressions for the four different ways in which the angles resulting from the intersection of X and Y may be considered: and the point of reference P is an infallible index as to the particular angle denoted by any one of the symbols. An instance of the *comprehensiveness* of the Notation has already been given in the statement that, *whatever be the position of* P, the bisectors of x y, y z, z x meet in a point. And the fact that, on this principle, no two straight lines which are not parallel can make equal angles with the same straight line, is an instance of the *precision* of the Notation.

The Angular and Linear Notation may also be applied to Solid Geometry. P will in this case denote the positive side of every plane under investigation, and the only condition with regard to its position is that none of the planes spoken of shall pass through it.

1263. (Proposed by N'IMPORTE.)—Dans un triangle quelconque ABC, on mène, à partir des sommets, trois droites dans des directions quelconques; ces trois droites par leurs intersections donnent naissance à un second triangle DEF. Par les sommets de celui-ci on fait passer des droites respectivement parallèles aux côtés du triangle ABC; on obtient ainsi un troisième triangle LMN. Démontrer que la surface du triangle DEF est moyenne proportionnelle entre les surfaces des deux autres.

Solutions (1) *by* W. HOPPS; J. MCDOWELL, B.A.; *and* S. WATSON; (2) *by* S. BILLS; *and others.*



1. Let Δ_1, Δ_2, Δ_3 denote the respective areas of the triangles ABC, DEF, LMN; and P_1, P_3 the perpendiculars from the vertices C, N of the triangles Δ_1, Δ_3 on the opposite sides AB, LM respectively. Join NA, NB, NC. Then by parallels we have

Δ NAC = EAC, and Δ NBC = DBC;

therefore $\quad \Delta_2 = \text{NAFB}.$

Hence $\quad \Delta_1 : \Delta_2 = P_1 : P_3$; also $\Delta_3 : \Delta_1 = P_3^2 : P_1^2$;

therefore $\quad \Delta_3 : \Delta_2 = P_3 : P_1 = \Delta_2 : \Delta_1$.

2. *Otherwise:* take A for origin of *Cartesian coordinates*, and AB, AC for the axes of x and y; and put $AB = r$, $AC = s$.

Let the equations of EF, FD, DE be respectively

$$x = my, \quad x - r = ny, \quad x = p(y - s).$$

Then the respective coordinates of D, E, F will be

$$\left(\frac{p(ns+r)}{p-n},\ \frac{ps+r}{p-n}\right),\ \left(\frac{mps}{p-m},\ \frac{ps}{p-m}\right),\ \left(\frac{mr}{m-n},\ \frac{r}{m-n}\right).$$

Hence the equations of LM, MN, NL are found to be

$$y = \frac{r}{m-n},\quad x = \frac{mps}{p-m},\quad \frac{x}{r} + \frac{y}{s} = \frac{p(ns+r)}{r(p-n)} + \frac{ps+r}{s(p-n)}.$$

Now $\Delta_1 = \frac{1}{2}rs \sin A$; also from the above we readily find

$$\Delta_2 = \frac{\left\{(p-m)\,r - ps\,(m-n)\right\}^2 \sin A}{2\,(m-n)\,(n-p)\,(p-m)},\quad \Delta_3 = \frac{\left\{(p-m)r - ps(m-n)\right\}^4 \sin A}{2rs\,(m-n)^2\,(n-p)^2\,(p-m)^2};$$

therefore $\Delta_1 \Delta_3 = \Delta_2^2$, or $\Delta_1 : \Delta_2 = \Delta_2 : \Delta_3$.

[As an exercise for our junior readers, we add a proof by *Trilinear Coordinates* :—

Let $\alpha=0,\ \beta=0,\ \gamma=0$ be the equations of the sides of the triangle DEF, and $a,\ b,\ c$ the lengths of these sides respectively. Then the triangle ABC has one of its vertices on each of the sides of the triangle of reference DEF (or on the prolongations of these sides); hence it may be readily shown (see *Educational Times* for January and April, 1854, Quest. 663) that the equations of the sides of the triangle ABC may be written

$$\alpha+n\beta+\frac{\gamma}{m}=0,\quad \frac{\alpha}{n}+\beta+l\gamma=0,\quad m\alpha+\frac{\beta}{l}+\gamma=0.$$

The equations of the sides of the triangle LMN, drawn through the vertices of the triangle of reference parallel to the sides of the triangle ABC, are

$$(na-b)\,\beta+\left(\frac{a}{m}-c\right)\gamma=0,\quad (lb-c)\,\gamma+\left(\frac{b}{n}-a\right)\alpha=0,$$

$$(mc-a)\,\alpha+\left(\frac{c}{l}-b\right)\beta=0.$$

From the foregoing equations of the sides of the triangles ABC, LMN, we find, by applying the formula proved in the Solutions of Quest. 1733 (*Reprint*, Vol. IV., pp. 53—56), that the areas of these triangles are

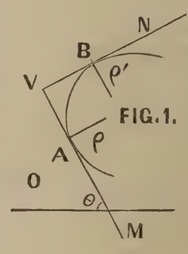
$$\Delta_1=\frac{abc\,(lmn-1)\,\Delta_2}{(lb-c)\,(mc-a)\,(na-b)},\quad \Delta_3=\frac{(lb-c)\,(mc-a)\,(na-b)\,\Delta_2}{abc\,(lmn-1)};$$

where Δ_2 is the area of the triangle of reference DEF.

Hence we have $\quad \Delta_1\Delta_3=\Delta_2^2$, or $\Delta_1:\Delta_2=\Delta_2:\Delta_3$].

1720. (Proposed by M. W. CROFTON, B.A.)—A given curve moves in its own plane, without rotation, so as always to pass through a fixed point A; in any of its positions draw a tangent at A, and a second tangent cutting this at right angles; and find the envelope of the latter.

Solution by the PROPOSER.



Let VA, VB (Fig. 1) be two perpendicular tangents to a given curve; let VA = T, VB = T′; let $\rho,\ \rho'$ be the radii of curvature at A, B; let θ = inclination of the tangent VA to a fixed axis, and let the curve be represented by the *intrinsic* equation $\rho=f(\theta)$. We shall have (see Quest. 1622, *Reprint*, Vol. III., p. 75)

$$\frac{d^2\mathrm{T}}{d\theta^2}+\mathrm{T}=\rho'-\frac{d\rho}{d\theta}.$$

Suppose now the right angle MVN to revolve, always touching the curve in two points; and in

each position let the curve be transferred without rotation so that the point of contact A shall fall upon a fixed point O (Fig. 2). The tangent VN will envelope some curve, and T, or OV, will be the perpendicular from the fixed point O on the tangent to the envelope. Now if R = radius of curvature of the envelope at P, we have, (since in any curve $\rho = \frac{d^2p}{d\theta^2}+p$)

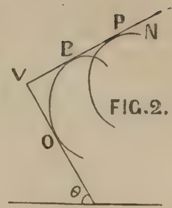


FIG. 2.

$$R = \frac{d^2T}{d\theta^2}+T=\rho'-\frac{d\rho}{d\theta}, \ \therefore R=f\left(\theta+\frac{\pi}{2}\right)-\frac{d}{d\theta}f(\theta).$$

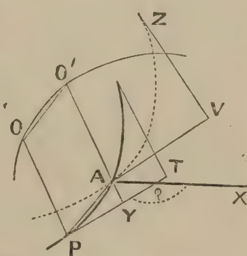
If ϕ = inclination of tangent of envelope to the axis, its intrinsic equation is therefore

$$R=f(\phi)-\frac{d}{d\phi}f\left(\phi-\frac{\pi}{2}\right).$$

This may be applied to various curves. If the moving curve is a circle, the envelope is an equal circle. If the moving curve is a cycloid, its equation will be $\rho = -4a\cos\theta$; and we shall find the equation of the envelope to be R=0, that is, a point; so that in this case the perpendicular tangent passes through a fixed point: a result which may be verified by other methods.

II. *Solution by* F. D. THOMSON, M.A.

1. Let AP, AZ be two positions of the moving curve which always passes through A and moves without rotation; then if P be the point which has moved up to A, O any other point which has moved to O′, AP is equal and parallel to OO′, and therefore AO′ is equal and parallel to PO. Hence any point O describes a curve about the fixed point A similar and equal to the original curve, but turned in an opposite direction; also the point O corresponds to A, and O′ to P on the original curve, and the tangent at O′ will be parallel to the tangent at P. Hence, in order to construct the tangent at A to the curve AZ, construct the locus of the point O, let O′ be the new position of O, and draw a line AV parallel to the tangent at O′.



2. Next to find the envelope of the tangent VZ perpendicular to AV. Draw AY perpendicular to PY. Then from the figure we see that AV = PY + perpendicular on tangent to AP which is perpendicular to PY. Hence if $p=f(\phi)$ be the equation to AP, we have $AV = PY+f(\phi-\frac{1}{2}\pi)=f'(\phi)+f(\phi-\frac{1}{2}\pi)$, since $PY=\frac{dp}{d\phi}=f'(\phi)$. Thus putting $AV=p'$, $\phi-\frac{1}{2}\pi=\phi'$, we have $p'=f'(\phi'+\frac{1}{2}\pi)+f(\phi')$; hence, dropping the accents, the tangential polar equation of the envelope is $p=f(\phi)+f'(\phi+\frac{1}{2}\pi)$.

Example. Let the curve be the equiangular spiral $r=ae^{\theta \cot \alpha}$, the fixed point A being the pole. Then $\phi=\theta+\alpha$, and $p=r \sin \alpha=a \sin \alpha\, e^{(\phi-)\alpha \cot \alpha}$; therefore $f'(\phi) = a \cos ae^{(\phi-\alpha) \cot \alpha}$; hence the equation of the envelope is

$$p = a\,(\cos \alpha\, e^{\frac{1}{2}\pi \cot \alpha} + \sin \alpha)\, e^{(\phi-\alpha) \cot \alpha},$$

which is that of an equiangular spiral.

1783. (Proposed by R. Tucker, M.A.)—Eliminate θ between each of the following sets of equations:

$$(\mathrm{I}) \;..\; \left\{\begin{array}{l} \mathrm{X} \cos(\theta-\mathrm{A}) + \mathrm{Y} \cos\theta = 2\mathrm{R} \sin(\theta+\mathrm{C}) \cos\theta \cos(\theta-\mathrm{A}) \ldots.\,(1) \\ \mathrm{X} \sin(\theta-\mathrm{A}) + \mathrm{Y} \sin\theta = -2\mathrm{R} \cos(\theta+\mathrm{C}) \sin\theta \sin(\theta-\mathrm{A}) \,..\,(2) \end{array}\right\};$$

$$(\mathrm{II}) \;..\; \left\{\begin{array}{l} \mathrm{Y}-\mathrm{X} \cot(\mathrm{B}+\tfrac{1}{2}\theta) = \mathrm{R}\left\{\cos \mathrm{C} - \sin(\mathrm{C}-\theta) \cot(\mathrm{B}+\tfrac{1}{2}\theta)\right\} \,..\,(1) \\ \mathrm{Y}+\mathrm{X} \tan(\mathrm{B}+\tfrac{1}{2}\theta) = \mathrm{R}\left\{\cos \mathrm{C} - \sin(\mathrm{C}-\theta) \tan(\mathrm{B}+\tfrac{1}{2}\theta)\right\} \,..\,(2) \end{array}\right\};$$

where A, B, C are the angles of a triangle, and R the radius of its circumscribing circle.

Solution by the Proposer; S. Bills; *and others.*

I. Here, from $\left\{(1) . \mathrm{X} \sin\theta + (2) . \mathrm{X} \cos\theta\right\}$ we obtain

$$\mathrm{X} \sin \mathrm{A} \cot 2\theta = x \cos \mathrm{A} + \mathrm{Y} - \mathrm{R} \sin \mathrm{B} \ldots\ldots\ldots\ldots\ldots (\alpha).$$

Again, from $\left\{(1) . \mathrm{X} \sin(\theta-\mathrm{A}) + (2) . \mathrm{X} \cos(\theta-\mathrm{A})\right\}$ we have

$$\cot 2\theta\,(\mathrm{X} \sin 2\mathrm{A} + \mathrm{Y} \sin \mathrm{A} - \mathrm{R} \sin \mathrm{C} \sin 2\mathrm{A}) = \mathrm{X} \cos 2\mathrm{A} + \mathrm{Y} \cos \mathrm{A} - \mathrm{R} \sin \mathrm{C} \cos 2\mathrm{A} \ldots.\,(\beta).$$

Eliminating θ between (α) and (β), we get

$$\mathrm{X}^2 + 2\mathrm{XY} \cos \mathrm{A} + \mathrm{Y}^2 - \tfrac{1}{2}(c+2b \cos \mathrm{A})\,\mathrm{X} - \tfrac{1}{2}(b+2c \cos \mathrm{A})\,\mathrm{Y} + \tfrac{1}{2}bc \cos \mathrm{A} = 0.$$

II. Eliminating X and Y successively, we get

$$\mathrm{Y}-\mathrm{R} \cos \mathrm{C} = -\mathrm{R} \sin(\mathrm{C}-\theta) \sin(2\mathrm{B}+\theta),\quad \mathrm{X} = \mathrm{R} \sin(\mathrm{C}-\theta) \cos(2\mathrm{B}+\theta);$$

$$\therefore \mathrm{X}^2 + (\mathrm{Y}-\mathrm{R} \cos \mathrm{C})^2 = \mathrm{R}^2 \sin^2(\mathrm{C}-\theta),\ \tan(2\mathrm{B}+\theta) = \mathrm{X}^{-1}(\mathrm{R} \cos \mathrm{C} - \mathrm{Y}) \,..\,(\gamma, \delta).$$

From this latter equation (δ), we have

$$\sin\theta = \frac{(\mathrm{Y}-\mathrm{R} \cos \mathrm{C}) \cos 2\mathrm{B} + \mathrm{X} \sin 2\mathrm{B}}{\left\{\mathrm{X}^2 + (\mathrm{Y}-\mathrm{R} \cos \mathrm{C})^2\right\}^{\frac{1}{2}}},$$

$$\cos\theta = \frac{(\mathrm{Y}-\mathrm{R} \cos \mathrm{C}) \sin 2\mathrm{B} - \mathrm{X} \cos 2\mathrm{B}}{\left\{\mathrm{X}^2 + (\mathrm{Y}-\mathrm{R} \cos \mathrm{C})^2\right\}^{\frac{1}{2}}}.$$

Substituting in (γ) there results

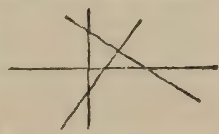
$$\mathrm{X}^2 + (\mathrm{Y}-\mathrm{R} \cos \mathrm{C})^2 = \mathrm{R}\left\{(\mathrm{Y}-\mathrm{R} \cos \mathrm{C}) \cos(\mathrm{A}-\mathrm{B}) - \mathrm{X} \sin(\mathrm{A}-\mathrm{B})\right\}.$$

[We may remark that the given equations represent the two feet-perpendicular lines in Quest. 1649 (*Reprint*, Vol. III., p. 58). In (I) the axes are AB, AC; in (II) they are lines through O parallel and perpendicular to AB. The result in each case is the nine-point circle of the given triangle, thus affording another proof of the theorem.]

On the Four-point and similar Geometrical Chance Problems.

By J. M. Wilson, M.A., F.G.S.

The four-line problem proposed by me as Quest. 1868 was to find the chance that if three lines were drawn at random in an infinite plane, a fourth line drawn at random would intersect the triangle formed by the other three.



If four lines are drawn, it will be found that two of them always intersect the triangle formed by the other three, and since the lines are equally at random the chance required is $\frac{1}{2}$. To put the solution in another form; if a blind-folded man were told that three lines were drawn on a large piece of paper, and was asked to draw a fourth, and to decide the probability of its intersecting the triangle; he might draw the line, and on examining the figure would see that if *his* line were one of two, it would intersect the triangle, and if not, it would not; and having nothing to guide him as to which was *his* line, would decide the probability to be $\frac{1}{2}$.

The four-point problem is to find the chance of four points at random in an infinite plane forming a reentrant quadrilateral. I argue thus:—Four lines at random determine three sets of four random points, which form one reentrant and two convex quadrilaterals, and therefore the chance is $\frac{1}{3}$.

This result follows from, and is as certain as, the axiom, that a random point may be looked on as the intersection of two random lines. And the reasoning may be justified as before. A blind-folded man is told there are four points on a sheet of paper, and is asked the chance of their being the angles of a reentrant quadrilateral. He requests that they may be joined by lines, so that each point has two and only two lines passing through it. The lines are produced indefinitely, and the only condition that depends on the space being supposed infinite is that the lines do intersect on the sheet of paper before him. He now examines the figure, and it is impossible for him to tell from which of the three quadrilaterals before him the four lines originated. And since only one, and always one of the three is reentrant, he must reason that the chance is $\frac{1}{3}$. (See the above Figure.)

I see here no uncertainty introduced by the space being infinite, and there is no comparison of infinities which is fallacious; it is equally true for this sheet of paper, if it is granted that the figure is such as can be drawn upon it.

Below, however, are some remarks made on this Solution by Dr. Ingleby, with whose judgment I have the misfortune to differ. He seems to me to have proved that the probability is less than $\frac{1}{2}$, by a method which is incapable ot assigning how much less. I can attach no meaning whatever to the last line, on which his whole argument depends.

On a Problem in the Theory of Chances.

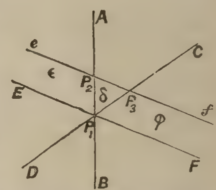
By C. M. Ingleby, LL.D.

Four points being placed at random in a plane, what is the chance of one of them falling within the triangle formed by the other three?

This problem has been variously solved by Professors Cayley, Sylvester, and Price; and above is a fourth solution by Mr. J. M. Wilson of Rugby, the Senior Wrangler of 1859. Strange to say, no two of these four results are alike. The value assigned by Mr. Wilson to the required chance is $\frac{1}{3}$.

Now I submit that, whatever be the true value, this at least is amenable to a very simple *reductio ad absurdum*.

If two points be taken at random, and three straight lines, lying in a plane passing through both, mutually intersect in one of them, the chance that the other will be found in one of the alternate angles α, β, γ is $\frac{\alpha+\beta+\gamma}{2(\alpha+\beta+\gamma)} = \frac{1}{2}$. This, at least, will not be disputed. We are indeed comparing two infinities, but two infinities of which we know all we need, viz., their ratio to each other. This result will be of use in dealing with the problem before us.



Take any two of the four points (P_1, P_2) and draw a straight line AB through both. Then the chance that P_3 (either of the other two points) lies out of AB may be taken as unity. Let it be to the right of AB. Through P_1, P_3 draw the straight line CD, and through P_2, P_3 draw the straight line *ef*. Also through P_1 draw the straight line EF parallel to *ef*. Let the area $EP_1P_2e = \epsilon$, $FP_1P_3f = \phi$, and $P_1P_2P_3 = \delta$. Now it is plain that in order that one of the four points may fall within the triangle formed by the other three, P_4 must fall within one of the areas $P_1P_2P_3$, AP_2e, CP_3f, DP_1B; that is, P_1 must lie within $P_2P_3P_4$, P_2 within $P_3P_4P_1$, P_3 within $P_4P_1P_2$, or P_4 within $P_1P_2P_3$. What is the chance of P_4 falling within one of these four areas? The chance of a point falling in AP_1E, BP_1D, or CP_1F, is, as we have seen, $\frac{\alpha+\beta+\gamma}{2(\alpha+\beta+\gamma)}$. Now conceive EF to move parallel to itself till it assumes the position *ef*. The effect of this transference is that the area AP_1E loses ϵ, and the area CP_1F loses ϕ, while δ is a gain to the favourable chances. Accordingly the chance of P_4 falling within one of the areas $P_1P_2P_3$, AP_2e, CP_3f, DP_1B, is $\frac{\alpha+\beta+\gamma-(\epsilon+\phi-\delta)}{2(\alpha+\beta+\gamma)} = \frac{1}{2} - \frac{\epsilon+\phi-\delta}{2(\alpha+\beta+\gamma)}$. The ratio $\frac{\epsilon+\phi-\delta}{2(\alpha+\beta+\gamma)}$ is $+$; for δ can only become infinite by either P_1P_2 or P_1P_3 or both becoming infinite; and even then $\epsilon+\phi$ is infinitely greater than δ. If any doubt be felt on this point, let it be considered that though P_2P_3 can be made greater than any assignable quantity, it cannot be made infinite by virtue of any position which can be assumed by AB and CD; for when the angle $AP_1C = \pi$, AB and CD coincide by the rotation of one or both those straight lines round P_1: but in this case one of those lines at least does not pass through any point in *ef*, which is contrary to the assumption.

We have then a positive magnitude $\epsilon+\phi-\delta$, which, in order that Mr. WILSON's value $\frac{1}{3}$ may hold good, must $= \frac{1}{3}(\alpha+\beta+\gamma)$; whereas $\frac{\alpha+\beta+\gamma}{\epsilon+\phi-\delta}$ may be made greater than 3, 4.... or any assignable magnitude.

1868. (Proposed by J. M. WILSON, M.A.)—Three straight lines are drawn at random on an infinite plane, and a fourth line is drawn at random to intersect them; find the probability of its passing through the triangle formed by the other three.

Solution by PROFESSOR WHITWORTH, M.A.

Of the four lines, two must and two must not pass within the triangle formed by the remaining three.

Since all are drawn at random, the chance that the last drawn should pass through the triangle formed by the other three, is consequently $\frac{1}{2}$.

The only case of exception will be when some of the lines are parallel; but the chance of parallelism is less than any assignable chance.

Hence the result $\frac{1}{2}$ remains correct.

1441. (Proposed by Dr. SALMON, F.R.S.)—A teetotum of n sides is spun an indefinite number of times, and the numbers turning up are added together; what is the chance that a given number s will be actually arrived at?

Solution by THOMAS BOND SPRAGUE, M.A.

A little consideration will show that the probability of s being arrived at in exactly i trials is the coefficient of x^s in $\left(\frac{x+x^2+x^3+\,..\,+x^n}{n}\right)^i$, or in H^i suppose. In the same way, the probability of s being arrived at in $(i+1)$ trials is the coefficient of x^s in H^{i+1}. Thus the total probability of s being arrived at in an indefinite number of trials is the coefficient of x^s in $\mathrm{H}+\mathrm{H}^2+\mathrm{H}^3+\,..\,+\mathrm{H}^i+\,..$, that is in $\frac{\mathrm{H}}{1-\mathrm{H}}$, or in $\frac{1}{1-\mathrm{H}}-1$, which is the same thing as the coefficient of x^s in $\frac{1}{1-\mathrm{H}}$. Substituting for H its value, the probability required is the coefficient of x^s in

$$\frac{n(1-x)}{n(1-x)-x(1-x^n)}, \text{ or in } \frac{n(1-x)}{n-(n+1)\,x+x^{n+1}}.$$

We must now expand this quantity in powers of x. For this purpose, writing for shortness' sake y for $\left(1-\frac{n+1}{n}x\right)$, we put it in the form

$$(1-x)\left(y+\frac{x^{n+1}}{n}\right)^{-1}=(1-x)\left(y^{-1}-\frac{x^{n+1}}{n}y^{-2}+\frac{x^{2n+2}}{n^2}y^{-3}-\ldots.\right).$$

The general term of this series is

$$(-1)^t(1-x)\frac{x^{tn+t}}{n^t}y^{-t-1}, \text{ or } (-1)^t(1-x)\frac{x^{tn+t}}{n^t}\left\{1+(t+1)\frac{n+1}{n}x+\,..\right.$$
$$\left.\ldots.+\frac{(t+1)\,(t+2)\ldots.(t+r)}{1\,.\,2\,.\,3\ldots.r}\left(\frac{n+1}{n}\right)^r x^r+\ldots.\right\};$$

and the coefficient of x^{tn+t+r} in this expansion is

$$\frac{(-1)^t}{n^t}\left\{\frac{(t+1)(t+2)\ldots(t+r)}{1.2.3\ldots r}\left(\frac{n+1}{n}\right)-\frac{(t+1)(t+2)\ldots(t+r-1)}{1.2.3\ldots(r-1)}\right\}\left(\frac{n+1}{n}\right)^{r-1}$$

$$=\frac{(-1)^t}{n^t}\cdot\frac{1.2.3\ldots(t+r-1)}{1.2\ldots t.1.2\ldots(r-1)}\cdot\frac{tn+t+r}{rn}\left(\frac{n+1}{n}\right)^{r-1}$$

$$=(-1)^t\cdot\frac{(r+1)(r+2)\ldots(r+t-1)}{1.2.3\ldots t}(tn+t+r)\frac{(n+1)^{r-1}}{n^{r+t}}.$$

If now we put $s=tn+t+r$, this is the coefficient of x^s, and is the general term of the probability required. Substituting in the last of the above forms for r its value $s-tn-t$, it becomes

$$(-1)^t\cdot\frac{(s-tn-t+1)(s-tn-t+2)\ldots\ldots(s-tn-1)}{1.2.3\ldots\ldots t}\cdot s\cdot\frac{(n+1)^{s-tn-t-1}}{n^{s-tn}}.$$

We must here give t all the values of which it is capable, viz., 0, 1, 2, .. T, the last being the integer quotient obtained on dividing s by $n+1$.

The required probability is thus found to be

$$\frac{(n+1)^{s-1}}{n^s}-s\cdot\frac{(n+1)^{s-n-2}}{n^{s-n}}+\frac{s(s-2n-1)}{1.2}\cdot\frac{(n+1)^{s-2n-3}}{n^{s-2n}}$$
$$-\frac{s(s-3n-1)(s-3n-2)}{1.2.3}\cdot\frac{(n+1)^{s-3n-4}}{n^{s-3n}}+\ldots\ldots$$

which agrees with the result obtained in a different manner by Mr. WOOLHOUSE, *Reprint*, Vol. I. p. 77.

1480. (Proposed by Professor SYLVESTER.)—Prove that if through the middle point of either diagonal of any of the three quadrilateral faces of a tetrahedral frustum, and the middle points of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will touch the same cone.

Solution by SAMUEL ROBERTS, M.A.

Complete the tetrahedron, and take as origin the apex so obtained, and as axes the edges meeting at that apex. The coordinates of the six angles of the frustum may be taken to be x_1, x_2; y_1, y_2; z_1, z_2. The plane passing through $(\frac{1}{2}x_2, \frac{1}{2}y_1, 0)$, $(0, \frac{1}{2}y_1, \frac{1}{2}z_1)$, $(\frac{1}{2}x_2, 0, \frac{1}{2}z_2)$ has for its equation

$$2(xy_1z_1+yz_2x_2+zx_2y_1)=x_2y_1(z_1+z_2).$$

We can at once write the equations of two other planes: viz., these are

$$2(xy_2z_2+yz_1x_1+zx_1y_2)=x_1y_2(z_1+z_2),\quad 2(xz_2y_2+yx_1z_2+zx_1y_1)=x_1z_2(y_1+y_2).$$

These planes intersect in the point

$$x=\frac{x_1x_2(y_2z_2-y_1z_1)}{2(K-L)},\quad y=\frac{y_1y_2(z_2x_2-z_1x_1)}{2(K-L)},\quad z=\frac{z_1z_2(x_2y_2-x_1y_1)}{2(K-L)},$$

where K, L are put for $x_2y_2z_2$, $x_1y_1z_1$.

The symmetry of the coordinates shows that the six planes drawn according to the conditions specified intersect in one point.

The planes may, therefore, be represented by

$$y_1z_1\mathrm{X}+x_2z_2\mathrm{Y}+x_2y_1\mathrm{Z}=0\ldots.(1),\qquad y_1z_2\mathrm{X}+x_2z_2\mathrm{Y}+x_1y_1\mathrm{Z}=0\ldots.(4),$$
$$y_2z_2\mathrm{X}+x_1z_1\mathrm{Y}+x_1y_2\mathrm{Z}=0\ldots.(2),\qquad y_1z_1\mathrm{X}+x_2z_1\mathrm{Y}+x_2y_2\mathrm{Z}=0\ldots.(5),$$
$$y_2z_1\mathrm{X}+x_1z_1\mathrm{Y}+x_2y_2\mathrm{Z}=0\ldots.(3),\qquad y_2z_2\mathrm{X}+x_1z_2\mathrm{Y}+x_1y_1\mathrm{Z}=0\ldots.(6);$$

and it is only necessary to show that, considered as Trilinear Coordinates, the coefficients of X, Y, Z represent points on a conic.

Forming an equation to the locus from (1), (2), (3), (4), we get

$$(xx_2-zz_1)(xx_1-zz_2)=p\left\{(xx_1+yy_1-zz_2)\,\mathrm{K}-(xx_2+yy_2-zz_1)\,\mathrm{L}\right\}$$
$$\times\left\{(yy_1+zz_1-xx_2)\,\mathrm{K}-(yy_2+zz_2-xx_1)\,\mathrm{L}\right\}\ldots\ldots.(a).$$

Now when $(y_1z_1,\ x_2z_1,\ x_2y_2)$, $(y_2z_2,\ x_1z_2,\ x_1y_1)$ are substituted for x, y, z in (a), we have the same results. Therefore the six sets of coefficients represent points on a conic, and reciprocally the six planes envelope a cone.

1481. (Proposed by Professor Hirst.)—Find the envelope of a conic which circumscribes a given triangle, and is cut harmonically by two fixed straight lines.

N.B.—A conic is said to be cut harmonically in two pairs of points, when the tangents at those points cut every other tangent harmonically, or, what amounts to the same thing, when the connectors of those points with any other point on the conic form a harmonic pencil.

I. *Solution by* H. R. Greer, B.A.

Granting that a conic through three fixed points may be represented by a line in such wise that to each point on the conic shall correspond one point on the line, and that the anharmonic ratio of four points on the conic shall be equal to that of the corresponding points on the line, the above problem may be reduced to one of a lower degree, so to speak. And, if this be solved, a retransformation of the result will be necessary in order to obtain the absolute solution of the question proposed.

Now, the possibility of this representation has been established by divers methods of geometrical derivation, and, notably, by that of Quadric Inversion, the fundamental principles of which may be thus summed up,—that being given three fixed points, "principal points," an arbitrary curve of the degree n not passing through any of the principal points will be transformed into one of the degree $2n$, passing n times through each of them; and this transformation (or, rather, representation) will be effected in such a manner that corresponding points shall connect homographically with the principal points. Hence an arbitrary line will be transformed into a conic passing through the three fixed points, anharmonic ratios being preserved. Loci of points and envelopes of curves will be transformed into the loci and envelopes of the corresponding points and curves.

Having recalled these principles to mind, the proposed problem may be replaced by the following:—Find the envelope of a line which is cut harmonically by two fixed conics. This envelope is a conic touching the four

common tangents of the two fixed conics. From which I conclude that the envelope in the proposed Question is a curve of the 4th degree, having double points at the vertices of the given triangle, and touching the four conics which can be described through these three vertices tangential to both the fixed lines.

II. *Solution by* SAMUEL ROBERTS, M.A.

Let a conic of the system be represented by $\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}=0$, and the two fixed lines by $L\alpha+M\beta+N\gamma=0$, $L'\alpha+M'\beta+N'\gamma=0$; then each of these lines is to pass through the pole of the other; hence we have

$$A^2LL'+B^2MM'+C^2NN'-2AB\,(LM'+L'M)-2BC\,(MN'+M'N)-2CA\,(NL'+N'L)=0.$$

The required envelope is obtained by taking, subject to this condition, the envelope of $\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}=0$, and the result is

$$\frac{1}{\alpha^2}(MN'-M'N)^2+\frac{1}{\beta^2}(NL'-N'L)^2+\frac{1}{\gamma^2}(LM'-L'M)^2$$
$$-\frac{2}{\alpha\beta}(MN'-M'N)(NL'-N'L)-\frac{2}{\beta\gamma}(NL'-N'L)(LM'-L'M)-\frac{2}{\gamma\alpha}(LM'-L'M)(MN'-M'N)=0.$$

1864. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$(1)..\ 1-n+\frac{n(n-1)}{1.2}..\pm\frac{n(n-1)..(n-r+1)}{1.2..r}=\pm\frac{(n-1)(n-2)..(n-r)}{1.2..r},$$

$$(2)..\ \frac{1}{m+1}..+\frac{1}{m+n}=\frac{n(n+1)..(n+m)}{1.2..m}\left\{\frac{1}{(m+1)^2}-\frac{n-1}{1}\cdot\frac{1}{(m+2)^2}+\&c.\right\}.$$

Solution by G. C. DE MORGAN, M.A.

1. Call the quantity on the left-hand side $\phi(r)$; then we have

$$\Delta\phi(r)=(-)^{r+1}.\frac{n(n-1)..(n-r)}{1.2..r(r+1)}=(-)^r.\frac{n(n-1)..(n-r)}{1.2..(r-1)}.\Delta\frac{1}{r}.$$

Applying the formula $\Sigma u_r\Delta v_r=u_rv_r-\Sigma v_{r+1}\Delta u_r$, we get

$$\phi(r)=(-)^r.\frac{n(n-1)..(n-r)}{1.2..r}-(n-1).\phi(r)+\text{a } constant,$$

therefore $$\phi(r)=(-)^r.\frac{(n-1)(n-2)..(n-r)}{1.2.3..r};$$

the constant being 0, as is found by making $r=1$.

2. The series on the left may be put in the form

$$-(m+1)(m+2)..(m+n)\cdot\frac{d}{dm}\cdot\frac{1}{(m+1)(m+2)..(m+n)},$$

or $$-\frac{(m+1)(m+2)..(m+n)}{1.2..(n-1)}\cdot\frac{d}{dm}(-\Delta)^{n-1}\frac{1}{m+1}.$$

Developing $(-\Delta)^{n-1}$ in powers of $E=1+\Delta$, and substituting $\frac{1}{m+2}$ for $E.\frac{1}{m+1}$, &c., we get

$$-\frac{(m+1)(m+2)..(m+n)}{1.2\,..(n-1)}\frac{d}{dm}\left\{\frac{1}{m+1}-\frac{n-1}{1}\cdot\frac{1}{m+2}+\&c.\right\}$$

$$=\frac{n(n+1)..(n+m}{1.2.3..m}\left\{\frac{1}{(m+1)^2}-\frac{n-1}{1}\cdot\frac{1}{(m+2)^2}+\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{(m+3)^2}-\&c.\right\}.$$

1885. (Proposed by the Rev. A. F. Torry, M.A.)—Investigate the following constructions for determining the point (T) of intersection of the common tangents of an ellipse and its circle of curvature at P. If O be the centre of the circle, C that of the ellipse, S either focus; then (1) T lies on the confocal hyperbola which passes through P; (2) OC bisects PT; and (3) SP, ST are equally inclined to OS.

Solution by the Proposer.

1. The first construction is deduced as follows. If P and Q be two points on a conic, the tangents at which meet in T; and if from P and Q tangents be drawn to a confocal conic, these form a quadrilateral in which can be inscribed a circle with centre T: moreover, the extremities of either of the other diagonals of this quadrilateral lie on another confocal conic. This may be proved by showing that the perpendiculars from T upon the sides of this quadrilateral are in a constant ratio to those drawn from T at right angles to the focal distances of P and Q. (See the articles by Mr. C. Taylor, in Nos. XI. and XII. of the *Messenger of Mathematics.*)

Now if an ellipse and a confocal hyperbola cut in P, and the tangent to the ellipse at P cut the hyperbola again in T; and if from a point on the hyperbola indefinitely near to P, and also from T, tangents be drawn to the ellipse, the inscribed circle of the quadrilateral so formed will be ultimately the circle of curvature at P.

The proposition enumerated above may be otherwise stated thus:—"If a quadrilateral be formed by drawing common tangents to an ellipse and a circle, the extremities of either of its diagonals lie on a confocal conic." Quest. 1814 is a particular case of the same proposition, two of the tangents coinciding in one of the sides of the triangle.

2. The second construction follows from the first if we remember that the centre of curvature is the pole of PT with respect to the confocal hyberbola.

It may also be proved independently. Considering, as before, that the ellipse and circle are inscribed in the same quadrilateral, three of whose sides ultimately coincide with the tangent PT; PT becomes a diagonal of the quadrilateral, and "the centres of all inscribed conics lie on a straight line bisecting the diagonals;" which proves the proposition. It is not difficult to deduce the first construction from the second by pure analysis.

3. The third construction is obtained by reciprocating with respect to either of the foci of the ellipse the well-known theorem, that "the common tangent of an ellipse and its circle of curvature at any point are equally inclined to the major axis of the ellipse."

1890. (Proposed by PROFESSOR CAYLEY.)—Find the equation of a conic passing through three given points and having double contact with a given conic.

Solution by the PROPOSER.

Let the given points be the angles of the triangle ($x = 0$, $y = 0$, $z = 0$), and let the equation of the given conic be $U = (a, b, c, f, g, h)(x, y, z)^2 = 0$; then the equation of the required conic is

$$U - (x\sqrt{a} + y\sqrt{b} + z\sqrt{c})^2 = 0,$$

for this is a conic having double contact with the conic $U = 0$, and, since the terms in (x^2, y^2, z^2) each vanish, it is also a conic passing through the given points.

It is clear that there are four conics satisfying the conditions of the Problem, viz., putting for shortness

$$P = x\sqrt{a} + y\sqrt{b} + z\sqrt{c}, \quad P_1 = -x\sqrt{a} + y\sqrt{b} + z\sqrt{c},$$
$$P_2 = x\sqrt{a} - y\sqrt{b} + z\sqrt{c}, \quad P_3 = \quad x\sqrt{a} + y\sqrt{b} - z\sqrt{c},$$

the four conics are $U - P^2 = 0$, $U - P_1{}^2 = 0$, $U - P_2{}^2 = 0$, $U - P_3{}^2 = 0$.

It may be remarked that the conics P, P have a fourth intersection lying on the line $y\sqrt{b} + z\sqrt{c} = 0$, and the conics P_2, P_3 a fourth intersection lying on the line $y\sqrt{b} - z\sqrt{c}$; which two lines are harmonics in regard to the lines $y = 0$, $z = 0$.

Similarly the conics P_1, P_2 have a fourth intersection on the line $x\sqrt{a} + z\sqrt{c} = 0$, and the conics P_1, P_3 a fourth intersection on the line $x\sqrt{a} - z\sqrt{c} = 0$; which two lines are harmonics in regard to the lines $z = 0$, $x = 0$. And the conics P_1, P_3 have a fourth intersection on the line $x\sqrt{a} + y\sqrt{b} = 0$, and the conics P_2, P_4 a fourth intersection on the line $x\sqrt{a} - y\sqrt{b} = 0$; which two lines are harmonics in regard to the lines $x = 0$, $y = 0$. It may further be remarked that the equations of any two of the four conics may be taken to be

$$\alpha yz + \beta zx + \gamma xy = 0, \quad \alpha' yz + \beta' zx + \gamma' xy = 0.$$

The general equation of a conic having double contact with each of these conics then is

$$n^2 z^2 - 2n(\gamma\alpha' + \gamma'\alpha) zy - 2n(\gamma\beta' + \gamma'\beta) zx - 4n\gamma\gamma' xy$$
$$+ [(\beta\gamma' - \beta'\gamma) x - (\gamma\alpha' - \gamma'\alpha) y]^2 = 0,$$

where n is arbitrary: and, having double contact with this conic, we have (besides the above-mentioned two conics) two new conics each passing through the angles of the triangle; viz., writing for greater convenience

$$n = \frac{(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)}{\mathrm{K}-\gamma\gamma'}, \quad \text{or} \quad \mathrm{K} = \gamma\gamma' + \frac{(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)}{n},$$

then the equations of the two new conics are

$$\gamma'\alpha\, yz + \gamma\beta'\, zx + \mathrm{K}xy = 0, \quad \gamma\alpha'\, yz + \gamma'\beta\, zx + \mathrm{K}xy = 0.$$

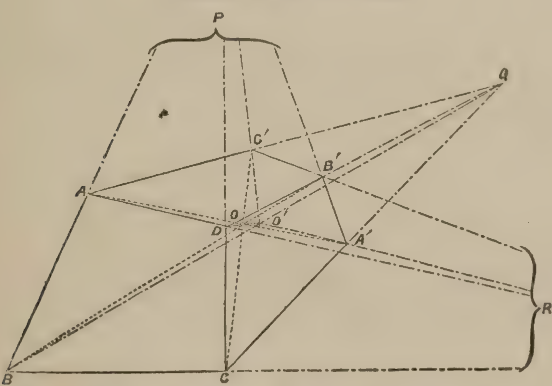
In fact, writing the equation under the form

$$\begin{aligned}
&[xz + (\beta\gamma'-\beta'\gamma)\,x + (\gamma\alpha'-\gamma'\alpha)\,y]^2 \\
&-4\,(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)\,xy - 4n\gamma\gamma'xy \\
&-2n\,(\beta\gamma'-\beta'\gamma)\,xz - 2n(\beta\gamma'+\beta'\gamma)\,xz \\
&-2n\,(\gamma\alpha'-\gamma'\alpha)\,yz - 2n\,(\gamma\alpha'+\gamma'\alpha)\,yz = 0,
\end{aligned}$$

we at once see that this is a conic having double contact with the conic $\gamma'\alpha\, yz + \gamma\beta'zx + \mathrm{K}xy = 0$, the equation of the chord of contact being $nz + (\beta\gamma'-\beta'\gamma)\,x + (\gamma\alpha'-\gamma'\alpha)\,y = 0$: and similarly it has double contact with the conic $\gamma\alpha'yz + \gamma'\beta zx + \mathrm{K}xy = 0$, the equation of the chord of contact being $nz - (\beta\gamma'-\beta'\gamma)\,x - (\gamma\alpha'-\gamma'\alpha)\,y = 0$.

1893. (Proposed by C. W. Merrifield, F.R.S.)—If the edges of any hexahedron meet four by four in three points, then the four diagonals meet in a point.

Solution by the Proposer.



The construction implies that the hexahedron should be the common frustum of three four-sided pyramids. Now if we consider the diagonal planes of these, that is to say, the planes through the vertices and the dia-

gonals of the bases, it is clear that the four diagonals of the hexahedron lie two by two on these six diagonal planes. Hence they pass through a point.

This demonstration assumes as a lemma a particular case of the following general proposition, which may be established inductively :—"If n lines lie two by two on $\frac{1}{2}n(n-1)$ planes, they all pass through a point." To establish this, begin with two lines in one plane, and add a line at a time. The proposition is then obvious.

II. *Solution by* J. R. ALLEN.

Let ABCDA′B′C′D′ be the given hexahedron, whose edges

BA, CD, A′B′, D′C′ meet in one point P,

AC′, DB′, CA′, BD′ meet in one point Q,

AD, C′B′, D′A′, BC meet in one point R;

it is required to show that the diagonals AA′, BB′, CC′, DD′ intersect in one point.

It is evident by Euc. xi. 2, that

AA′, BB′ intersect in plane PBA′	BB′, CC′ intersect in plane RC′B
AA′, CC′ intersect in plane QAC	BB′, DD′ intersect in plane QDB
AA′, DD′ intersect in plane RAD′	CC′, DD′ intersect in plane PCD′
therefore AA′ cuts BB′, CC′, DD′	CC′ cuts AA′, BB′, DD′
BB′ cuts AA′, CC′, DD′	DD′ cuts AA′, BB′, CC′.

And it is evident that if out of four straight lines every one intersects the three remaining ones, they must all intersect in one point.

III. *Solution by the* REV. R. TOWNSEND, M.A.

By homographic transformation of the figure, the three points of concurrence of the three different quartets of connectors of different pairs of adjacent vertices may be sent to infinity, in which case the four connectors of the four pairs of opposite vertices pass evidently through the centre of the parallelepiped into which the hexahedron becomes then transformed. Or, without any transformation, the four latter lines pass evidently, in every case, through the pole of the plane determined by the three points of concurrence of the three former quartets, with respect to any quadric passing through the eight vertices of the figure.

As a hexahedron reciprocates into an octohedron, the reciprocal property, that when, of an octohedron, the three different quartets of intersections of different pairs of adjacent faces are each complanar, the four intersections of the four pairs of opposite faces are also complanar, appears in the same manner from the reciprocal consideration, that the four latter lines lie evidently, in every case, in the polar plane of the point determined by the three planes of complanarity of the three former quartets, with respect to any quadric touching the eight faces of the figure.

1895. (Proposed by the Rev. R. TOWNSEND, M.A.)—Two circles A and B, whose radii are a and b, touch at two points P and Q a common circle

whose radius is r; show that the length of their common tangent (AB), external or internal according as their contacts with it are of similar or opposite species, is given by the formula $(\mathrm{AB}) = \dfrac{\sqrt{(r \pm a) \cdot (r \pm b)}}{r} \cdot (\mathrm{PQ})$; and hence prove immediately the following extension of Ptolemy's Theorem given by Mr. Casey. When four circles A, B, C, D touch a common circle, the six common tangents, AB, &c., of their six groups of two external or internal according as the contacts of the two with the common circle are similar or opposite, are connected by the relation

$$(\mathrm{BC}) \cdot (\mathrm{AD}) + (\mathrm{CA}) \cdot (\mathrm{BD}) + (\mathrm{AB}) \cdot (\mathrm{CD}) = 0.$$

N.B.—By supposing the radius of one of the four circles, D, to be $= 0$, Mr. Casey has obtained immediately from this relation the following equation for a pair of conjugate circles touching the remaining three A, B, C; viz.,

$$(\mathrm{BC}) \cdot \sqrt{(\alpha)} + (\mathrm{CA}) \cdot \sqrt{(\beta)} + (\mathrm{AB}) \cdot \sqrt{(\gamma)} = 0\,;$$

where $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the equations of A, B, and C.

Solution by A. Renshaw; W. H. Laverty; S. W. Bromfield; J. H. Taylor, B.A.; *and many others.*

1. If A′, B′, are the centres of the two circles, we have

$$\frac{2r^2 - (\mathrm{PQ})^2}{2r^2} = \cos \mathrm{O} = \frac{(r \pm a)^2 + (r \pm b)^2 - (\mathrm{A'B'})^2}{2(r \pm a)(r \pm b)},$$

therefore $$\frac{\mathrm{PQ}^2}{r^2} = \frac{(\mathrm{A'B'})^2 - (a \pm b)^2}{(r \pm a)(r \pm b)} = \frac{(\mathrm{AB})^2}{(r \pm a)(r \pm b)}\,;$$

whence $$r \cdot (\mathrm{AB}) = \sqrt{(r \pm a)(r \pm b)} \cdot (\mathrm{PQ}).$$

2. If the radii of the four circles be a, b, c, d, and P, Q, R, S the points of contact, we have, by (1) and Ptolemy's theorem (Euc. VI., D),

$$(\mathrm{BC}) \cdot (\mathrm{AD}) + (\mathrm{CA}) \cdot (\mathrm{BD}) + (\mathrm{AB}) \cdot (\mathrm{CD}) =$$
$$\frac{\sqrt{(r \pm a)(r \pm b)(r \pm c)(r \pm d)}}{r^2} \left\{ (\mathrm{QR}) \cdot (\mathrm{PS}) + (\mathrm{RP}) \cdot (\mathrm{QS}) + (\mathrm{PQ}) \cdot (\mathrm{RS}) \right\} = 0.$$

3. When the radius of the circle D vanishes, the points D and S coincide, and may be anywhere on the circumference of the circle (O say) which touches the circles A, B, C: moreover if $\alpha = 0$ be the equation of the circle A, the length of the tangent from any point on O to A will be $\sqrt{(\alpha)}$ (Salmon's *Conics*, Art. 90); hence the relation in (2) becomes

$$(\mathrm{BC}) \cdot \sqrt{(\alpha)} + (\mathrm{CA}) \cdot \sqrt{(\beta)} + (\mathrm{AB}) \cdot \sqrt{(\gamma)} = 0.$$

1901. (Proposed by R. Tucker, M.A.)—Find the curve whose circle of curvature always passes through a fixed point.

I. *Solution by* Professor Mannheim.

Taking the given point as origin, the inverse of the required curve must be a straight line, otherwise it could not have three-pointic contact with all

the straight lines inverse to the circles of curvature of the primitive. This primitive, therefore, must necessarily be itself a circle passing through the given point.

II. *Solution by the* PROPOSER; J. H. TAYLOR, B.A.; *and others.*

If α, β be the coordinates of the centre of curvature, and p, q stand for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, respectively, we have

$$\alpha = x - \frac{p(1+p^2)}{q}, \quad \beta = y + \frac{1+p^2}{q},$$

and the equation to the circle of curvature will be

$$(X-x)^2 + 2(X-x)\frac{p(1+p^2)}{q} + (Y-y)^2 - 2(Y-y)\frac{1+p^2}{q} = 0.$$

(i.) When the circle always passes through a fixed point, taking this point for origin, we have

$$x^2+y^2 = 2x\frac{p(1+p^2)}{q} + 2y\frac{1+p^2}{q};$$

which leads to a circle passing through the given point, and the equation to the circle of curvature is $X^2+Y^2 = 2\alpha X + 2\beta Y$.

(ii.) When the circle always touches a fixed line, take this line for one of the axes, and we have $\alpha=\rho$ or $\beta=\rho$, where ρ is the radius of curvature; and these equations lead to a circle touching the fixed line.

1925. (Proposed by J. GRIFFITHS, M.A.)—Given four points on a circle whose radius is r; show that the centroids (centres of gravity of the areas) of the four triangles that can be formed from them lie on another circle, whose radius is $\frac{1}{3}r$.

Solution by F. D. THOMSON, M.A.; W. H. LAVERTY; R. TUCKER, M.A.; *the* REV. J. L. KITCHIN, M.A.; *and others.*

If A, B, C, D be the four points, and A′, B′, C′, D′ the respective centroids of the triangles BCD, CDA, DAB, ABC, it is clear that the line A′B′ is parallel to AB and equal to $\frac{1}{3}$AB; for A′ and B′ lie respectively on the lines joining B and A with the middle point (M suppose) of CD, and

$$MA' : MB = MB' : MA = 1 : 3.$$

Thus the sides of the quadrilateral A′B′C′D′ are parallel to those of ABCD, and equal to one-third of those sides in linear magnitude; whence the truth of the theorem is obvious.

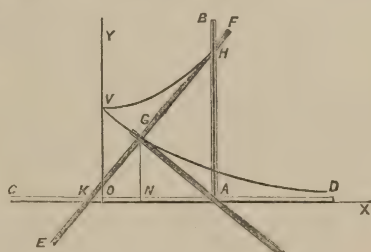
[Mr. TOWNSEND remarks that the property in question is a particular case of the following:—

If A, B, C, D, E, F, &c. be the positions of any number (n) of equal masses distributed in any manner in space, O that of their centre of gravity,

and A′, B′, C′, D′, E′, F′, &c. those of the centres of gravity of their n different groups of $(n-1)$; then always the two systems of n points A, B, C, D, E, F, &c. and A′, B′, C′, D′, E′, F′, &c. are similar, oppositely placed with respect to each other, have O for their centre of similitude, and have $(n-1) : 1$ for their ratio of similitude. The truth of this is evident, for the several lines AA′, BB′, CC′, DD′, EE′, FF′, &c. all connect through O, and are there divided internally in the common ratio of $(n-1) : 1$.]

1801. (Proposed by W. H. LAVERTY.)—A thin bar of wood AB with a groove running along the middle has a perpendicular arm CD projecting each way; and a similar piece EGAF is connected with it by a pivot at A. In EGAF there is a small sharp-edged wheel at G, turning on a pivot which lies in the plane of EGAF, and in the direction of EGF. A pencil is inserted in the concourse of both grooves at H; and CAD being slid along the axis of X, two curves are generated. Find them, and show that the curve described by the pencil is the evolute of that described by the wheel.

Solution by the PROPOSER.



Here AG $(=a)$, which is constant in length, is a tangent to the lower curve, which is therefore the *Tractory*, whose equation (if $ON=x$, and $NG=y$) is

$$x = -(a^2-y^2)^{\frac{1}{2}} + a\log\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}.$$

Again, let $OA=x_1$, $AH=y_1$; then

$$x_1 = ON+NA = x+(a^2-y^2)^{\frac{1}{2}} = a\log\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y};$$

but $y = y_1 \cdot \dfrac{KN\,.\,NA}{KA\,.\,AN} = y_1 \cdot \dfrac{y^2}{a^2}$; therefore $y = \dfrac{a^2}{y_1}$;

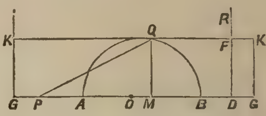
therefore $$x_1 = a\log\frac{y_1+(y_1^2-a^2)^{\frac{1}{2}}}{a}.$$

This is the equation to the *Catenary*; and it is the evolute to the Tractory. See Price's *Calculus*, Vol. I., Fig. 106.

[The construction in the Question seems to have been first given by Sir John Leslie, in his *Geometry of Curve Lines*, p. 398. A fuller description of the instrument is given by Davies (*Hutton's Course*, 12th ed., Vol. II., p. 519), who proposes to call it *Leslie's Catenarian Trammel*.]

1700. (Proposed by T. T. WILKINSON, F.R.A.S.)—The line DR is perpendicular to the diameter AB of a given semicircle AQB; it is required to find in the circumference a point Q such that, if we join Q with P, a point anyhow given in the line DA, and draw QF perpendicular to DR, the sum or difference of PQ and QF may be given.

Solution by the PROPOSER; E. McCORMICK; H. MURPHY; E. FITZGERALD; J. DALE; *and many others.*

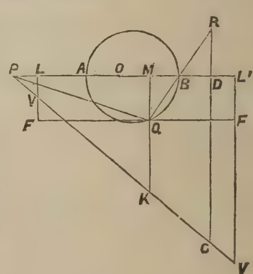


Suppose Q determined; in MP take MG = PQ, and let GK perpendicular to, and QK parallel to, PD meet in K. Now QF = MD by parallels; and MG = PQ; or, PQ + QF = GM + MD = GD = a given line. Hence the point G and the perpendicular GK are given by position. Hence, we have only to draw PQ to a point Q in the circumference of the circle such that PQ = the perpendicular QK, and this has already been done in Quest. 219 of the *Key* by various correspondents.

In like manner, when the *difference* is given: take DG on the contrary side of D, equal to the given difference; draw the perpendicular GK; and PQ, making PQ = QK or MG, as in the above. Hence this is evidently also a case of Question 219, as before.

1764. (Proposed by T. T. WILKINSON, F.R.A.S.)—The same being supposed as in Quest. 1700; it is required to determine Q, so that if QR be drawn making any given angle with DR, the sum, or difference, of PQ and QR may be given.

Solution by the PROPOSER; E. McCORMICK; H. MURPHY; E. FITZGERALD; J. DALE; *and many others.*



Draw DC perpendicular to DP; also PC making $\angle$ DCP = the given angle. In CP, towards P, take CV = the given sum; draw LVF perpendicular to PD. Now determine the point Q, by the construction in Question 219 of the *Key*, making PQ : QF (a perpendicular upon LV) = PC : PD; the required point is Q.

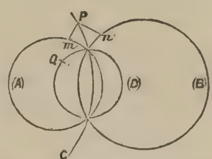
Draw QR making the given angle with DR; and let QM, a perpendicular to PD, meet CP at K. Then, because of equal angles and parallels, DP : PC = DM : QR = DM : CK; whence QR = CK. Also, PQ = QF = PQ : ML = PC : PD = VK : LM; whence PQ = VK. Therefore PQ + QR = VK + KC = VC = the given *sum*.

When the *difference* is given; take CV, on the contrary side of C, equal

to that difference; and draw PQ and QF as before. Then, in the demonstration, it follows that $KC = QR$, $KV = QP$; and therefore $PQ - QR = KV - KC = CV =$ the given difference.

1655. (Proposed by M. W. CROFTON, B.A.)—Let the equations of two circles (A) and (B), whose radii are r and r', be $\Theta = 0$ and $\Theta' = 0$; then the two circles (C) and (D), whose equations are $\frac{\Theta}{r} - \frac{\Theta'}{r'} = 0$ and $\frac{\Theta}{r} + \frac{\Theta'}{r'} = 0$, intersect at right angles.

Solution by the PROPOSER.



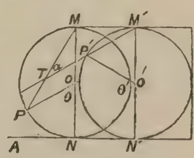
Take a point P on (C), infinitely near the intersection; then it is known that putting the coordinates of P for (x, y) in Θ and Θ' we have $\Theta = 2r \cdot Pm$, $\Theta' = 2r' \cdot Pn$; hence $Pm = Pn$, so that the circle (C) bisects the angle of $\Theta = 0$, $\Theta' = 0$.

Also if we took a point Q on (D), infinitely near the intersection, we should have $-\Theta = 2r \cdot \delta$, $\Theta' = 2r' \cdot \delta'$, where δ and δ' are the evanescent perpendiculars from Q on (A) and (B); hence the circle (D) bisects the other angle of $\Theta = 0$, $\Theta' = 0$.

Therefore the circles (C) and (D) cut at right angles.

1758. (Proposed by M. W. CROFTON, B.A.)—If two tangents to a cycloid include a constant angle, show that their sum has a constant ratio to the included arc of the curve.

Solution by J. DALE; E. MCCORMICK; *and others.*



Let NPM, N′P′M′ be two positions of the generating circle corresponding to the points P, P′ on the cycloid; and let θ, θ' be the auxiliary angles; then PM, P′M′ (intersecting in T) are tangents at P and P′, and the included arc of cycloid $= 2(PM - P'M')$. Also, $\alpha = \angle MTM' = \frac{1}{2}(\theta' - \theta)$, $MM' = NN' = a(\theta' - \theta)$, and in the triangle MTM′,

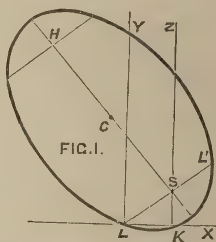
we have $TM = MM' \cdot \sin MM'T \cdot \operatorname{cosec} MTM' = \alpha \operatorname{cosec} \alpha \cdot P'M'$;

therefore $TP = PM - \alpha \operatorname{cosec} \alpha \cdot P'M'$, so also $TP' = \alpha \operatorname{cosec} \alpha \cdot PM - P'M'$;

therefore $TP + TP' = (1 + \alpha \operatorname{cosec} \alpha)(PM - P'M') = (1 + \alpha \operatorname{cosec} \alpha)$ arc PP'.

1583. (Proposed by R. TUCKER, M.A.) — A system of similar ellipses passes through a fixed point which coincides with one end of their parameters; find the locus of their centres, the envelope of the other parameters, and also that of the system when the locus of the focus is (1) a straight line, (2) a circle.

Solution by the PROPOSER.



Take L (Fig. 1) for fixed extremity of parameter, and for axis in (1), LX at right angles to the given line KZ, and LY parallel to it. Suppose LK $= k$, e the given eccentricity and (α, β) the coordinates of the centre; then if $\angle$SLK $= \theta$, we have

$$\alpha = k - ae \sin\theta,$$
$$\beta = k \tan\theta + ae \cos\theta,$$
$$k = a\,(1-e^2)\cos\theta;$$

whence eliminating a and θ, we have

$$(1-e^2)^2\,\alpha + e\,(1-e^2)\,\beta = k\,(1-e^2+e^4) \,..\, (A);$$

the equation to a straight line cutting KZ at a distance $ke\,(1-e^2)^{-1}$ from K equal to half the projection of the line between the foci on KZ.

Now the equation to the parameter through H, a point whose coordinates are $(k-2ae\sin\theta,\ k\tan\theta+2ae\cos\theta)$ is

$$y - x\tan\theta = \frac{2ke}{1-e^2}\sec^2\theta,$$

the envelope of which is

$$x^2 = \frac{8ke}{1-e^2}\left(-y+\frac{2ke}{1-e^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),$$

indicating a parabola whose focus is L, axis YL, and parameter $\dfrac{8ke}{1-e^2}$.

If (r, ϕ) be any point on the ellipse, we have

$$r\left\{1-e^2\sin^2(\phi-\theta)\right\} = 2a\,(1-e^2)\left\{\cos(\phi-\theta)+e\sin(\phi-\theta)\right\}$$
$$= 2k\left\{\cos\phi + e\sin\phi + \tan\theta\,(\sin\phi - e\cos\phi)\right\}.$$

Differentiating and arranging the equations, we have

$$\tan^3\theta\left\{2k\,(\sin\phi - e\cos\phi)\right\} + \tan^2\theta\left\{2k\,(\cos\phi+e\sin\phi) - r + e^2r\cos^2\phi\right\}$$
$$+\tan\theta\left\{2k\,(\sin\phi - e\cos\phi) - 2e^2r\sin\phi\cos\phi\right\} + 2k\,(\cos\phi + e\sin\phi)$$
$$-r\,(1-e^2\sin^2\phi) = 0,$$

$$\tan^2\theta\,(e^2r\sin\phi\cos\phi) + e^2r\tan\theta\cos 2\phi + k\,(\sin\phi - e\cos\phi) - e^2r\sin\phi\cos\phi = 0;$$

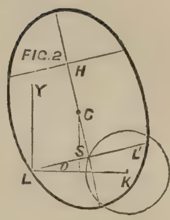
i. e., we have to eliminate $z\ (=\tan\theta)$ between

$$az^3 + bz^2 + 2cz + d = 0, \quad a'z^2 + b'z + c = 0,$$

where $2\,(a'+c) = a$, and $b - b' = d$; whence we get

$$(b-b')^2\,a'^3 - 2c\,(b-b')\,(b-2b')\,a'^2 + \left[b^2c^2 + b^2\,(b-b')\left\{b'\,(b-2b') + 2c^2\right\}\right]a'$$
$$+ 2c\left\{c^2\,(2b'^2 - bb' + 2c^2) - b'^3\,(b-b')\right\} = 0 \ldots\ldots\ldots\ldots\ldots (C),$$

where the letters have the above values, and the resulting curve is the required envelope.



Next (Fig. 2) let K be the centre of the given circle, and the axes LK and a line LY at right angles to it.

Then, if c = the given radius, we have

$$\alpha = a\,(1-e^2)\cos\theta - ae\sin\theta,$$
$$\beta = ae\cos\theta + a\,(1-e^2)\sin\theta,$$
$$c^2 = a^2(1-e^2)^2 + k^2 - 2ak\,(1-e^2)\cos\theta\,;$$

whence, eliminating a and θ, we obtain

$$(\alpha^2+\beta^2)(1-e^2)^2 - 2k\,(1-e^2)^2\alpha - 2k\,(1-e^2)\,e\beta$$
$$= (c^2-k^2)(1-e^2+e^4)\,\ldots\ldots\,(\mathrm{D}),$$

or the locus of centres is a circle.

The equation to the parameter through the point H, whose coordinates are $\left\{a\,(1-e^2)\cos\theta - 2ae\sin\theta\right\}$, $\left\{2ae\cos\theta + a\,(1-e^2)\sin\theta\right\}$, is

$$y - x\tan\theta = 2ae\sec\theta = \frac{2ek}{1-e^2} \pm \frac{2e\sec\theta}{1-e^2}\,(c^2-k^2\sin^2\theta)^{\frac{1}{2}};$$

whence $\quad (c^2-k^2)\left\{y - \dfrac{2ek}{1-e^2}\right\}^2 \left\{x^2(1-e^2)^2 - 4e^2(c^2-k^2)\right\}$

$$= c^2\left\{x^2(1-e^2) + 4e^2(c^2-k^2)\right\}^2 \,\ldots\ldots\ldots\,(\mathrm{E})$$

is the equation to the envelope.

In the particular case when L is on the given circle (or $c=k$), we have

$$y - x\tan\theta = 2ae\sec\theta = \frac{4ke}{1-e^2}\,\ldots\ldots\ldots\ldots\ldots\ldots\,(\mathrm{E}'),$$

showing that the parameters pass through a fixed point on the tangent at L.

The equation to the ellipse in the general case leads to a complicated result, but in the above case it becomes

$$r\left\{1-e^2\sin^2(\phi-\theta)\right\} = 4k\left\{\cos(\phi-\theta) + e\sin(\phi-\theta)\right\}\cos\theta$$
$$= 2k\left\{\cos\phi + \cos(\phi-2\theta) + e\sin\phi + e\sin(\phi-2\theta)\right\}.$$

Differentiating, we obtain

$$re^2\sin(2\phi-2\theta) = 4k\left\{\sin(\phi-2\theta) - e\,\mathrm{ccs}\,(\phi-2\theta)\right\};$$

and as a final result we obtain for the equation to the envelope sought

$$(1-e^2)\,r^4 - 4kr^2\left\{r\cos\theta\,(1-e^2) + er\sin\theta + k\,(1+e^2)\right\}$$
$$+\,4k^2(r\cos\theta + er\sin\theta)^2 = 0\,\ldots\ldots\ldots\ldots\ldots\ldots\,(\mathrm{F}).$$

Similar results hold in each case for the curves below the line LK.

On Approximation to a Curvilinear Area.

By Professor De Morgan.

The brevity of the following makes a problem of it. If y_0, y_1, &c., be ordinates of which the abscissæ 0, h, $2h$, &c., differ by h, and if $\mathrm{E}\phi(x) = \phi(x+1)$, we know that $\int_0^{nh} y_x dx$ is $h\,(\log \mathrm{E})^{-1}$ applied to $y_n - y_0$. If we

can find a function $h\psi\mathrm{E}\,.(\mathrm{E}^k-1)^{-1}$, in which, when $\mathrm{E}=1+\Delta$, a number of terms of the development in powers of Δ agree with those of $h(\log\mathrm{E})^{-1}$, we may from it construct an approximation to the integral, and find an approximation to the error committed. But, to preserve the symmetry which appears in the two modes of representation of $(\log\mathrm{E})^{-1}$, $\psi\mathrm{E}\,.(\mathrm{E}^k-1)^{-1}$, or $\chi\Delta$, should become $-\chi(-z)$ when $z(1-z)^{-1}$ is written for z.

The function $\frac{1}{2}(\mathrm{E}+1)(\mathrm{E}-1)^{-1}$ satisfies the conditions, and, applied to y_n-y_0, it gives $\frac{1}{2}y_0+y_1+\ldots.+y_{n-1}+\frac{1}{2}y_n$, the *surveyor's term*, as I always call it, of the ordinary method of quadratures. But the surveyors, or at least the engineers, have advanced a step, and have occasion to use *Simpson's Rule.* It is got by making $2n$ subdivisions, and considering the arcs from y_0 to y_2, from y_2 to y_4, &c., as arcs of parabolas with axes parallel to y. It is the next case of the method; and $\psi\mathrm{E}$ is $\mathrm{E}^2+4\mathrm{E}+1$.

If P_k abbreviate $\Delta^k y_{2n-k}+(-1)^k\Delta^k y_0$, the development of $\chi\Delta$ is

$$(\tfrac{1}{2}y_0+\ldots.+\tfrac{1}{2}y_{2n})-\tfrac{1}{12}\mathrm{P}_1-\tfrac{1}{24}\mathrm{P}_2-\tfrac{1}{48}\mathrm{P}_3-\tfrac{1}{96}\mathrm{P}_4-\ldots.$$

The development of $(\log\mathrm{E})^{-1}(y_{2n}-y_0)$ has $-\frac{19}{720}\mathrm{P}_3-\frac{3}{160}\mathrm{P}_4$.
And $h\psi\mathrm{E}\,(\mathrm{E}^2-1)^{-1}(y_{2n}-y_0)$ gives

$$\tfrac{1}{3}h\,(y_0+4y_1+2y_2+4y_3+2y_4+\ \ldots.\ +4y_{2n-1}+y_{2n}),$$

which is Simpson's rule. It may be corrected by subtracting

$$h\left(\frac{19}{720}-\frac{1}{48}\right)\mathrm{P}_3+h\left(\frac{3}{160}-\frac{1}{96}\right)\mathrm{P}_4,$$

or
$$\frac{h}{60}\left\{\frac{\Delta^3 y_{2n-3}-\Delta^3 y_0}{3}+\frac{\Delta^4 y_{2n-4}+\Delta^4 y_0}{2}\right\}.$$

The functions ψ will be best found by drawing the parabolas $a+bx$, $a+bx+cx^2$, &c., through two, three, &c. consecutive points. The correction may be best investigated, I think, in the preceding way. But as the error of Simpson's rule is of the fourth order, I suppose the engineers will not want a biquadratic parabola for a few years to come.

The coefficients of $\left\{\log(1+\Delta)\right\}^{-1}$, so far as usually given, are

$$1,\ \frac{1}{2},\ -\frac{1}{12},\ \frac{1}{24},\ -\frac{19}{720},\ \frac{3}{160},\ -\frac{863}{60480}.$$

The three next, whether ever before printed I know not, are

$$\frac{275}{24192},\ -\frac{33953}{3628800},\ \frac{8183}{1036800}.$$

1521. (Proposed by J. M. Wilson, M.A., F.G.S.) — Show that, in a geometric progression of an odd number of terms, the arithmetic mean of the odd numbered terms is greater than the arithmetic mean of the even-numbered terms, if the common ratio be any positive rational quantity not equal to unity.

Solutions by C. M. INGLEBY, LL.D.

Let the given series be $1+a+a^2+\ \ldots.\ +a^{2n}$, and suppose throughout that a is not equal to unity. Then we have to show that

$$\frac{1+a^2+a^4+\ \ldots.\ +a^{2n}}{a+a^3+\ \ldots.\ +a^{2n-1}} > \frac{n+1}{n}.\quad \text{Write this } \alpha_n > \frac{n+1}{n};\ \text{then}$$

$$\alpha_{n+1}+\frac{1}{\alpha_n} = \frac{1+a^2+a^4+\ldots.+a^{2n+2}}{a+a^3+\ \ldots\ldots\ +a^{2n+1}} + \frac{a+a^3+\ \ldots +a^{2n-1}}{1+a^2+a^4+\ \ldots\ +a^{2n}}$$

$$= \frac{1+2\,(a^2+a^4+\ \ldots.\ +a^{2n})+a^{2n+2}}{a+a^3+\ \ldots.\ +a^{2n+1}}$$

$$= \frac{(1+a^2+a^4+\ldots.\ +a^{2n})\,(1+a^2)}{(1+a^2+a^4+\ \ldots.\ +a^{2n})\,a} = a+\frac{1}{a}\ \ldots\ldots\ldots\ldots(1).$$

Now $a+\dfrac{1}{a} > 2$, that is, $> \dfrac{n+2}{n+1}+\dfrac{n}{n+1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$

therefore $\quad \alpha_{n+1} = \left(a+\dfrac{1}{a}\right)-\dfrac{1}{\alpha_n} > \dfrac{n+2}{n+1}+\left(\dfrac{n}{n+1}-\dfrac{1}{\alpha_n}\right)\ \ldots\ldots..(3).$

But if $\alpha_n > \dfrac{n+1}{n}$, $\dfrac{n}{n+1} > \dfrac{1}{\alpha_n}$, and $\left(\dfrac{n}{n+1}-\dfrac{1}{\alpha_n}\right)$ is positive;

therefore, by (3), $\quad \alpha_{n+1} > \dfrac{n+2}{n+1}$, if $\alpha_n > \dfrac{n+1}{n}$.

But $\alpha_1 = a+\dfrac{1}{a} > \dfrac{2}{1}$; $\therefore\ \alpha_2 > \dfrac{3}{2}$, $\alpha_3 > \dfrac{4}{3}$, &c., and generally $\alpha_n > \dfrac{n+1}{n}$.

2. *Otherwise:* Let $\alpha_n = \dfrac{1+a^2+a^4+\ \ldots.\ +a^{2n}}{a+a^3+\ ..\ +a^{2n-1}} = \dfrac{a^{2n+2}-1}{a^{2n+1}-a}$;

then, converting this into a continued fraction, we get

$$\alpha_n = \left(a+\frac{1}{a}\right)-\cfrac{1}{\left(a+\frac{1}{a}\right)-\cfrac{1}{\left(a+\frac{1}{a}\right)-\&c.}}\ \ldots\ldots\ldots\ldots(4).$$

By simple inspection of (4) we obtain the relation (1), viz., $\alpha_{n+1} = a+\dfrac{1}{a}-\dfrac{1}{\alpha_n}$, and the rest of the proof follows as before.

[Other Solutions are given in the *Reprint*, Vol. III., p. 15.]

1554. (Proposed by Professor CAYLEY.)—Show that, in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are

For the circle of maximum curvature .... $y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$,
for the ellipse .... $y_2 = a(1-e^2)$,
for the circle of minimum curvature .... $y_3 = \dfrac{a\left\{(1-e^2+e^4)^{\frac{1}{2}}-e^2\right\}}{(1-e^2)^{\frac{1}{2}}}$,
and that these values are in the order of increasing magnitude.

Solution by the REV. J. L. KITCHIN, M.A.; E. MCCORMICK; *and others.*

Let $b^2x^2+a^2y^2 = a^2b^2$ be the equation to the ellipse; then, if ρ be the radius of curvature at any point, we easily get, by substitution in the usual formula, $a^2b^2\rho^2 = (a^2-e^2x^2)^3$. Now ρ is a minimum (ρ_1, say) when $x=a$, and a maximum (ρ_2, say) when $x=0$;

therefore $\qquad \rho_1 = a(1-e^2)$, and $\rho_2 = \dfrac{a^2}{b} = \dfrac{a}{(1-e^2)^{\frac{1}{2}}}$.

The equations of the circles corresponding to ρ_1, ρ_2, that is to say, of the circles of maximum and minimum curvature, are respectively

$$(x-ae^2)^2+y^2 = a^2(1-e^2)^2, \quad x^2+\left(y+\frac{ae^2}{(1-e^2)^{\frac{1}{2}}}\right)^2 = \frac{a^2}{1-e^2}.$$

If in the equations of these circles, and in that of the ellipse, we put $x=ae$, we readily obtain the values of the ordinates given in the Question.

Now $y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$, and $y_2 = a(1-e)(1+2e+e^2)^{\frac{1}{2}}$; $\therefore\ y_2>y_1$.

Again, $y_3>y_2$, if $\left\{(1-e^2+e^4)^{\frac{1}{2}}-e^2\right\}^2 > (1-e^2)^3$, or if

$$1-e^2+e^4-2(1-e^2+e^4)^{\frac{1}{2}}+1>0, \text{ or if } \left\{(1-e^2+e^4)^{\frac{1}{2}}+1\right\}^2>0;$$

hence $y_3>y_2$, and therefore $y_3>y_2>y_1$.

1820. (Proposed by the Rev. J. BLISSARD.)—Prove that

$$\frac{m^m}{x+m}-\frac{m}{1}\cdot\frac{(m-1)}{x+m-1}+\frac{m(m-1)}{1.2}\cdot\frac{(m-2)^m}{x+m-2}-\&c. = \frac{1.2\ldots.m.x^{m-1}}{(x+1)(x+2)\ldots(x+m)}.$$

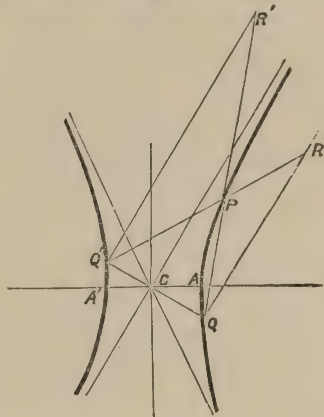
Solution by SAMUEL ROBERTS, M.A.

This is the direct result of applying the usual formula for the decomposition of rational Fractions to the right-hand member.

1836. (Proposed by R. TUCKER, M.A.)—Through the extremities of a diameter of an hyperbola (or its conjugate) at right angles to one asymptote,

straight lines are drawn parallel to the other; if the straight lines joining the extremities of the diameter to any point on the curve be produced, they will intercept on the parallels portions whose difference is constant.

Solution by the PROPOSER; E. MCCORMICK; *and others.*



Let A, A′ be the vertices of the hyperbola, and QCQ′ a diameter at right angles to one of the asymptotes. Connect P, any point on the curve, with Q, Q′; and produce QP, Q′P to meet the parallels to the asymptote (through Q′, Q) in R′, R; then shall the difference between Q′R′ and QR be constant.

For let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the equation to the hyperbola referred to the axes; then, transferring the axes of reference to CQ and the asymptote, and observing that $\tan\theta = \tan \text{ACQ} = \frac{a}{b}$, the equation becomes

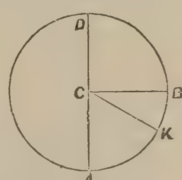
$$\frac{4cxy}{c^2 - x^2} = \frac{2ab}{c},$$

where $c = \text{CQ} = \frac{b}{\sqrt{(e^2-2)}}$.

Hence, since the difference of Q′R′ and QR $= \frac{4cxy}{c^2-x^2}$, the property is proved.

1848. (Proposed by M. W. CROFTON, B.A.)—Supposing the density of the population of the metropolitan area (radius 8 miles) to vary inversely as the distance from the centre, find the probability of two persons taken at random living nearer than 8 miles to each other.

Solution by the PROPOSER.



If an indefinite number of radii diverge at equal angular intervals from the centre of a circle, and a series of points are uniformly distributed along these radii, it is clear that the density of these points varies inversely as the distance from the centre.

It is easily seen that the required probability is unaltered by restricting one of the points to the fixed radius AC; we may also restrict the other to the *semicircle* ABD.

Make the angle ACK $= \frac{1}{3}\pi$; and draw CB perpendicular to AD. From

the solution to Quest. 1838 (*Reprint*, Vol. V., p. 53) we see that if the second point is on a radius within the sector ACK, the probability $p=1$; if within KCB, $p=\dfrac{3\theta-\pi}{2\sin\theta}+2\cos\theta$; if within BCD, $p=\dfrac{\pi-\theta}{2\sin\theta}$, where θ is the inclination of the radius to CA. Hence the required probability is

$$p=\frac{1}{3}+\frac{1}{\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left(\frac{3\theta-\pi}{2\sin\theta}+2\cos\theta\right)d\theta+\frac{1}{\pi}\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi-\theta}{2\sin\theta}\,d\theta.$$

But
$$\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi-\theta}{\sin\theta}\,d\theta=\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta},$$

$$\therefore\ p=\frac{1}{3}-\frac{1}{2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{d\theta}{\sin\theta}+\frac{2}{\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\cos\theta\,d\theta+\frac{3}{2\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}+\frac{1}{2\pi}\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta};$$

$$\text{or, } p=\frac{1}{3}-\frac{1}{4}\log 3+\frac{2}{\pi}\left(1-\frac{\sqrt{3}}{2}\right)+\frac{1}{2\pi}\left(\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}+3\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}\right).$$

To find series for the above definite integrals, let $t=\tan\frac{1}{2}\theta$; then

$$\int\frac{\theta\,d\theta}{\sin\theta}=2\int\frac{dt}{t}\tan^{-1}t=2\left(t-\frac{t^3}{3^2}+\frac{t^5}{5^2}-\frac{t^7}{7^2}+\&c.\right);$$

therefore $\displaystyle\int_{0}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\sin\theta}=2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\&c.\right)$; also

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\theta\,d\theta}{\text{in}\,\theta}=2\left(1-\frac{1}{3^2}+\&c.\right)-\frac{2}{\sqrt{3}}\left(1-\frac{1}{3^2}\cdot\frac{1}{3}+\frac{1}{5^2}\cdot\frac{1}{3^2}-\&c.\right).$$

The series $1-\dfrac{1}{3^2}+\dfrac{1}{5^2}-\&c.$ converges very slowly, but by means of certain artifices, which we have not space to give here, its value may be found without trouble to be ·9159; also

$$1-\frac{1}{3^2}\cdot\frac{1}{3}+\frac{1}{5^2}\cdot\frac{1}{3^2}-\frac{1}{7^2}\cdot\frac{1}{3^3}+\&c.=\cdot 96677.$$

Hence

$$p=\frac{1}{3}-\frac{1}{4}\log 3+\frac{2}{\pi}\left(1-\frac{\sqrt{3}}{2}\right)+\frac{1}{\pi}\left(\cdot 9159+3\times\cdot 9159-\cdot 96677\times\sqrt{3}\right),$$

which will give for the required probability $p=\cdot 7771$, nearly.

1957. (Proposed by the Rev. R. Townsend, M.A.)—Show that the chords of quickest and slowest descent from the highest point of an ellipse in a vertical plane are at right angles to each other and parallel to the axes of the curve.

I. *Solution by* ARTHUR COHEN, B.A.; E. MCCORMICK; *and others.*

Take the normal and tangent at the highest point P of the ellipse as the axes of x and y respectively. Then the equation to the ellipse is

$$ax^2 + bxy + cy^2 + dx = 0.$$

Let r be the length of a chord through P making an angle θ with the normal; then putting $r\cos\theta$ for x, and $r\sin\theta$ for y, we have

$$r(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta) + d\cos\theta = 0.$$

Therefore $\dfrac{r}{\cos\theta}$, which evidently is proportional to the square of the time of descent down the chord, equals $\dfrac{-d}{a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta}$.

Now if we transfer the origin to the centre, the equation to the ellipse becomes

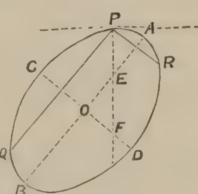
$$ax^2 + bxy + cy^2 + f = 0;$$

and if we denote by ρ the semi-diameter making an angle θ with the normal at P, we have, by putting $\rho\cos\theta$, $\rho\sin\theta$, for x and y,

$$\rho^2(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta) + f = 0.$$

Hence it follows that $\dfrac{r}{\cos\theta} = \dfrac{d}{f}\rho^2$, and therefore $\dfrac{r}{\cos\theta}$ is a maximum and a minimum when ρ is a maximum and a minimum, that is, when ρ is the major axis and the minor axis; hence the time of descent is a maximum and minimum when the chord r is parallel to the axes of the ellipse.

II. *Solution by* ARCHER STANLEY; M. COLLINS, B.A.; H. MURPHY; *the* REV. J. L. KITCHIN, M.A.; T. J. SANDERSON, B.A.; J. DALE; *the* PROPOSER; *and many others.*



Let the vertical through the highest point P of the ellipse cut the axes AB, CD in E and F. Now PR being drawn perpendicular to AB, a circle with centre E and radius PE will lie wholly *within* the ellipse and touch it at R as well as at P, since the curve is symmetrical about AB. But this being the case, PR is obviously the chord of quickest descent. Similarly the circle whose centre is F and radius PF will not only touch the ellipse in P, but also in Q, the image of P relative to CD; and since this circle lies wholly *without* the ellipse, PQ is the chord of slowest descent.

In the same manner evidently it may be shown that the chords of swiftest and slowest descent from the highest point of an ellipsoid anyhow situated, are parallel respectively to the least and greatest axes of the surface.

1965. (Proposed by H. R. GREER, B.A.)—Four conics through four points form a harmonic system; prove that if two conjugates be a circle and an equilateral hyperbola, the other two must be of equal eccentricities.

Solution by Archer Stanley.

Let S and Σ be the conjugate conics which pass through the intersections of the equilateral hyperbola H with the circle C, and form with these a harmonic system; that is to say, A being any one of the four common intersections, let the tangents at A to S and Σ be harmonic conjugates relative to the tangents at A to H and C.

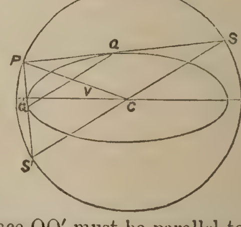
Now, by a well known theorem, the pairs of rays are in involution which are drawn through A parallel to the asymptotes of the several conics passing through the intersections of H and C; and the chords of the arcs which these pairs intercept on C are concurrent; they form, in fact, a pencil of rays homographic with the pencil of tangents at A.

Hence h, c, s, σ, the chords of the arcs of C intercepted by parallels through A to the asymptotes of H, C, S, Σ, form a harmonic pencil. But h is manifestly a diameter of C, and c is at infinity, consequently s and σ are parallel to and equidistant from h. The chords s and σ, therefore, subtend supplemental angles at A, that is to say, the asymptotes of S are inclined to each other at the same angles as are the asymptotes of Σ; which proves the theorem.

N.B.—The system of conics includes two parabolas, which likewise form, with H and C, a harmonic system; their axes are parallel to the connectors of A with the extremities of the diameter perpendicular to h, and consequently are at right angles to each other and inclined at an angle of 45° to the asymptotes of H. They are likewise parallel to the asymptotes of the rectangular hyperbola which is the locus of the centres of all the conics of the system.

1968. (Proposed by A. Renshaw.)—If from any point P in a circle concentric with a given ellipse, and the radius of which is equal to the distance between the ends of the major and minor axes, a pair of tangents be drawn to the ellipse and produced to meet the circle in the points S and S′; prove that the line SS′ is parallel to the polar of P.

Solution by T. J. Sanderson, B.A.; *the* Rev. R. H. Wright, M.A.; H. Tomlinson; *the* Rev. J. L. Kitchin, M.A.; *the* Proposer; *and others.*

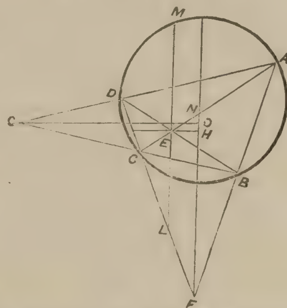


By a well known theorem, the circle concentric with a given ellipse, and of radius equal to the distance between the ends of the major and minor axes, is the locus of the intersection of pairs of tangents to the ellipse which cut at right angles. Hence SPS′ is a right angle; therefore SS′ is a diameter of the circle, and consequently passes through C the centre of the ellipse.

Let QQ′ be the polar of P, and join PC meeting QQ′ in V. Then QQ′ is bisected in V and SS′ in C by the same straight line PC. Hence QQ′ must be parallel to SS′.

1950. (Proposed by Professor SYLVESTER.)—If A, B, C, D be four points in a circle; and if AB, CD produced meet in F, and AD, BC produced meet in G, prove that the lines which bisect the angles F and G are at right angles to each other.

Solution by J. McDOWELL, F.R.A.S.; W. L. BOWDITCH; C. LAW; H. MURPHY; A. COHEN; S. W. BROMFIELD; H. TOMLINSON; W. M. MADDEN; J. DALE; T. J. SANDERSON, B.A.; Rev. R. HARLEY, F.R.S.; M. COLLINS, B.A.; T. COTTERILL, M.A.; *and many others.*



This may be at once deduced from the following theorem, which was set as a question at Christ's College, Cambridge, in 1859.

"ABCD is a quadrilateral inscribed in a circle; AC, BD meet in E; AB, CD produced meet in F; and AD, BC in G; show that the lines bisecting the angles AED, AFD are parallel to one another, and also those bisecting the angles AEB, AGB."

Let FN bisect the angle AFD and meet AC in N, GO bisect the angle AGB and meet FN in O, EH bisect AEB and LM bisect AED.

Since the angles ABD and ACD are equal, therefore the angles FBE and FCE are equal; therefore FCE, CEL and half AFD are together equal to two right angles, and therefore also to FCE, CEL and CLE. Therefore CLE equals CFN, and therefore LM and FN are parallel. Similarly EH and GO are parallel. Therefore EO is a parallelogram, and its opposite angles GOF and MEH are equal, but MEH is obviously a right angle, therefore GOF is a right angle.

II. *Solution by* H. McCOLL.

Before applying the principles of Angular and Linear Notation to prove Professor Sylvester's theorem, I will make a few additions to my former article on the subject. (*Reprint*, Vol. V., p. 74.)

DEF.—Let Z represent any curve, and p any point in the same plane; then z will denote the perpendicular from p upon the curve Z. Interpretation:—Let any tangent touch the curve at the point t; then z denotes the length of the line pt *provided that* pt *is perpendicular to the tangent.* The perpendicular z is understood to be *positive* when p and the point of reference are *on the same side* of the curve, and *negative* when *on opposite* sides of it.

1. In accordance with this definition and previous conventions, whatever be the position of the point of reference we shall have as the equation to the circle

$$(z) = (x-x')^2 + (y-y')^2 + 2\,(x-x')\,(y-y')\cos \mathrm{xy} - (z' \sin \mathrm{xy})^2,$$

in which X and Y are any straight lines not parallel, Z the circumference of the circle, and x', y', z' the perpendiculars upon X, Y, Z respectively from the centre.

2. Let p be any point and X, Y, Z any straight lines in the same plane; let x, y, z denote respectively the perpendiculars from p upon X, Y, Z; and

let x', y', z' denote respectively the perpendiculars upon X, Y, Z from the opposite intersections YZ, ZX, XY. Then, whatever be the position of the point of reference P, we shall have

$$x \sin \mathrm{yz} + y \sin \mathrm{zx} + z \sin \mathrm{xy} = x' \sin \mathrm{yz} = y' \sin \mathrm{zx} = z' \sin \mathrm{xy},$$

and therefore $\quad -z \sin \mathrm{xy} = x \sin \mathrm{yz} + y \sin \mathrm{zx} - z' \sin \mathrm{xy}$.

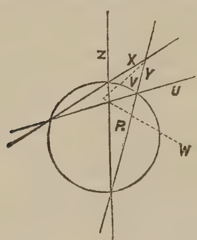
3. If the right-hand side of the last equation vanishes, the left-hand side must also vanish. This can only happen on the hypothesis that $z = 0$ or that $\sin \mathrm{xy} = 0$. The first hypothesis evidently restricts the position of p to the line Z; and the second, since it leaves z indeterminate, places no restriction whatever on the position of p. This last fact I overlooked, when I gave $(z) = x \pm y$ as a *necessary* consequence of the hypothesis $\sin \mathrm{xy} = 0$.

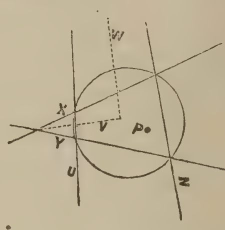
4. When the line Z bisects the angle xy we shall have $(z) = ax - ay$, in which a is any constant.

5. Let $(v) = ax + by + c$ and $(w) = a_1x + b_1y + c_1$ be the equations to V and W respectively. The straight lines V and W are parallel when $ab_1 = a_1b$, and at right angles when $aa_1 = -bb_1$.

We now proceed to the demonstration of Professor SYLVESTER's theorem, which may be otherwise enunciated as follows :—

If X, Y, Z, U be any straight lines whose intersections xz, zy, yu, ux are on the circumference of the same circle; then the bisectors, V and W say, of xy and zu are parallel or at right angles according to the position of the point of reference. The latter case will result from the position of P (the point of reference) in the subjoined figures.





Let p be any point in the plane; x, y, z, u its perpendicular distances from X, Y, Z, U respectively; and z', u' the perpendicular distances of the intersection xy from Z and U respectively. Then from (2) we get

$$-z \sin \mathrm{xy} = x \sin \mathrm{yz} + y \sin \mathrm{zx} - z' \sin \mathrm{xy} \;\ldots\ldots\ldots\ldots (A);$$

and $$-u \sin \mathrm{xy} = x \sin \mathrm{yu} + y \sin \mathrm{ux} - u' \sin \mathrm{xy} \;\ldots\ldots\ldots\ldots (B).$$

But from the known relations between angles in the same segment of a circle and angles in opposite segments, we have $\sin \mathrm{yu} = -\sin \mathrm{zx}$, and $\sin \mathrm{ux} = -\sin \mathrm{yz}$. Substituting, therefore, in (B) and subtracting (B) from (A), we get

$$u \sin \mathrm{xy} - z \sin \mathrm{xy} = x\,(\sin \mathrm{yz} + \sin \mathrm{zx}) + y\,(\sin \mathrm{yz} + \sin \mathrm{zx})$$
$$+ (u' - z') \sin \mathrm{xy}.$$

If for convenience we denote the coefficient of u and z by m, that of x and y by n, and the other constant $(u' - z') \sin \mathrm{xy}$ by c, the last equation becomes

$$mu - mz = nx + ny + c.$$

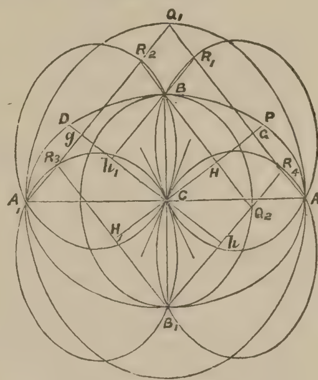
So far, no restriction has been put on the position of p; if now we restrict p

to the line W, the coordinates u and z become equal, and $mu-mz$ vanishes, so we get $(w) = mu-mz = nx+ny+c$.

Similarly if p be restricted to the line V, we get $(v) = ax-ay$; and since the coefficients of x and y in (v) and (w) satisfy the criterion in (5) the lines V and W are at right angles.

1922. (Proposed by W. GODWARD.)—Let AA_1, BB_1 be the major and minor axes of an ellipse, and CP, CD any pair of semi-conjugate diameters; draw AG, BH, B_1H_1 perpendicular to CP, and A_1g, B_1h, Bh_1 perpendicular to CD; also let AG, A_1g meet in Q_1; BH, B_1h in Q_2; AG, Bh_1 in R_1; A_1g, BH in R_2; A_1g, B_1H_1 in R_3; and AG, B_1h in R_4. Prove that the sum of the areas of the loci of Q_1, Q_2 is equal to the sum of the areas of the loci of R_1, R_2, R_3, R_4.

Solution by the PROPOSER; S. W. BROMFIELD; *the* REV. J. L. KITCHIN, M.A.; *and others.*



Let m be the tangent of the $\angle$ PCA, then, by conics, $\tan DCA = -\dfrac{b^2}{a^2m}$.

We have also the coordinates of

$A\ldots(a, 0)$; $A_1\ldots(-a, 0)$;

$B\ldots(0, b)$; $B_1\ldots(0, -b)$.

From which we at once obtain the equations to

$$AG\ldots y = -\frac{1}{m}(x-a)\ldots\ldots\ldots(1);$$

$$A_1g\ldots y = \frac{a^2m}{b^2}(x+a)\ldots\ldots\ldots(2);$$

$$BH\ldots y-b = -\frac{1}{m}x\ \ldots\ldots\ldots(3);$$

$$B_1h\ldots y+b = \frac{a^2m}{b^2}x\ldots\ldots\ldots\ldots(4);$$

$$B_1H_1\ldots y+b = -\frac{1}{m}x\ldots\ldots\ldots(5);\quad Bh_1\ldots y-b = \frac{a^2m}{b^2}x\ldots\ldots\ldots\ldots(6).$$

Eliminating m, by direct multiplication, from (1) and (2), (3) and (4), (1) and (6), (2) and (3), (2) and (5), and (1) and (4) respectively, we have the equations of the loci of the following points, viz.,

$Q_1\ldots a^2x^2+b^2y^2 = a^4\ldots\ldots\ldots\ldots\ldots\ldots(7)$; $\quad Q_2\ldots a^2x^2+b^2y^2 = b^4\ \ldots\ldots\ldots\ldots(8)$

$R_1\ldots a^2x^2+b^2y^2-a^3x-b^3y = 0\ \ldots(9)$; $\quad R_2\ldots a^2x^2+b^2y^2+a^3x-b^3y = 0\ (10)$

$R_3\ldots a^2x^2+b^2y^2+a^3x+b^3y = 0\ldots(11)$; $\quad R_4\ldots a^2x^2+b^2y^2-a^3x+b^3y = 0\ (12)$.

We learn from (7) that the locus of Q_1 is an ellipse concentric with the primitive one, that its semi-axes are $\dfrac{a^2}{b}$ and a, and the area $= \dfrac{a^3}{b}\pi$, the major axis being perpendicular to AA_1. From (8) that the locus of Q_2 is also a concentric ellipse whose semi-axes are b and $\dfrac{b^2}{a}$ and area $= \dfrac{b^3}{a}\pi$, the major

axis being perpendicular to AA_1. And from (9), (10), (11), and (12) that the loci of R_1, R_2, R_3 and R_4 are also ellipses passing through (A, C, B), (A_1, C, B), (A_1, C, B_1), and (A, C, B_1) respectively, that their major axes are all perpendicular to AA_1, and that the several coordinates of their centres are $\left(\frac{a}{2}, \frac{b}{2}\right)$, $\left(-\frac{a}{2}, \frac{b}{2}\right)$, $\left(-\frac{a}{2}, -\frac{b}{2}\right)$, and $\left(\frac{a}{2}, -\frac{b}{2}\right)$; also that the semi-axes of each are $\frac{\sqrt{(a^4+b^4)}}{2b}$ and $\frac{\sqrt{(a^4+b^4)}}{2a}$, and area $= \frac{a^4+b^4}{4ab}\pi$.

It hence follows that the sum of the areas of the loci of Q_1 and Q_2 is $= \frac{a^4+b^4}{ab}\pi =$ the sum of the areas of the loci of R_1, R_2, R_3 and R_4.

Cor. 1.—Subtracting (9) from (11), we have $y = -\frac{a^3}{b^3}x$, the equation of the common tangent of the loci of R_1 and R_3. Also subtracting (10) from (12), we have $y = \frac{a^3}{b^3}x$ the equation to the common tangent of R_2 and R_4. It hence appears that the loci of R_1 and R_3, and likewise the loci of R_2 and R_4, have simple contact at C.

Cor. 2.—The rectangle of the major or of the minor axes of the ellipses which are the loci of Q_1 and Q_2, is equal to the square on the major or on the minor axis of the primitive ellipse.

Correction of an inaccuracy in Dr. Ingleby's Note on the Four-point Problem.

In my Note on this problem (*Reprint*, Vol. V., p. 81), I committed an inaccuracy which I now ask leave to correct. I wrote, "This problem has been variously solved by Professors Cayley, Sylvester, and Price." There seem to have been five distinct solutions. (1) That of Professor Sylvester: this was founded on a private communication from Professor Cayley, and was published by the former without the authority of the latter. This solution gave $\frac{1}{4}$. Professor Sylvester has seen reason to withdraw his acquiescence in this result, as virtually implying that infinite extent is bounded by a convex line, and Professor Cayley is probably in the same position. (2) That of Professor De Morgan, which gave $\frac{1}{2}$. (3) That founded on a principle employed by Mr. Woolhouse in the analogous question of three points in a plane forming an acute-angled triangle. This solution gives $\frac{35}{12\pi^2}$. (4) Mr. Wilson's solution, which gives $\frac{1}{3}$. (5) Another which gives $\frac{3}{8}$.

The problem arose in a sort of theory of points subordinate to Professor Sylvester's method of Compound Partitions, and was originally propounded by him in one of his lectures on Partitions, delivered at King's College. Professor Sylvester has been supposed to intimate his belief that the value of the probability in question (whether or not within assignable limits) is essentially indeterminate.

Mr. Wilson is right in saying of my Note that it seems to show that the probability is less than $\frac{1}{2}$, by a method which is incompetent to determine

how much less. My very object was to show that the value is $\frac{1}{2}$ minus a positive quantity that is *indeterminate.*

Note on Dr. Ingleby's Strictures on Mr. Wilson's Solution of a Problem in Chances.

By the Rev. Professor Whitworth.

Assuming Dr. Ingleby's figure and notation, we agree with him that the required chance in any configuration of the points is expressed by $\frac{1}{2} - \frac{\epsilon + \phi - \delta}{2(\alpha + \beta + \gamma)}$, where $\epsilon + \phi - \delta$ is a variable quantity dependent upon the configuration of the first three points.

The required probability will therefore be $\frac{1}{2} - x$, where x is a mean value of the variable fraction $\frac{\epsilon + \phi - \delta}{2(\alpha + \beta + \gamma)}$. In order to show that Mr. Wilson's solution is wrong, it would be necessary to show that the proper mean x cannot be $\frac{1}{6}$. Dr. Ingleby, however, only shows that the variable fraction may have values less than any assignable value, a fact by no means inconsistent with the supposition that its mean value is $\frac{1}{6}$.

In fact we agree with Mr. Wilson that Dr. Ingleby seems to have proved that the probability is less than $\frac{1}{2}$ by a method which is incapable of assigning how much less.

A Solution of the Problem of determining the probability that four points taken at random in a plane shall form a re-entrant quadrilateral.

By G. C. De Morgan, M.A.

If four points be taken at random upon a hyperbola, the probability that they shall form a re-entrant quadrilateral is $\frac{1}{2}$. For this will be the case if one point fall upon one branch and the other three on the other, and not otherwise; and there are 8 ways in which this may happen, and 16 is the whole number of ways of distributing the points between the two branches. Now the hyperbola whose equation is $x^2 + axy - y^2 + bx + cy + d = 0$ may, by taking the coefficients properly, be made to pass through any set of four points; but we cannot in general pass two such hyperbolas through the same set. Hence we may consider all the different possible sets of four points as being made up of collections lying on the different hyperbolas represented by the equation above.

Since, whatever hyperbola the four fall on, the chance of a re-entrant quadrilateral is $\frac{1}{2}$, the chance on the supposition that they are taken anywhere in the plane, is $\frac{1}{2}$.

The cases where the four points are the intersections of two of the hyperbolas are not here taken into account, the chance that they shall be so being infinitely small. For, wherever three of them may fall, all such hyperbolas (in other words, all rectangular hyperbolas) passing through them intersect again in the same point, and the chance that the fourth point shall fall upon this intersection is infinitely small.

The same result is obtained by supposing each set of four points to consist of two pairs, separating all the possible pairs of points into collections lying on all possible different straight lines, and considering these straight lines two and two together.

NOTE ON QUESTION 1837. BY J. GRIFFITHS, M.A.

If we apply this theorem to any obtuse-angled triangle ABC and its self-conjugate circle, we have the following result. Let x be any point on one of the sides of the triangle (BC for instance), and let x be joined to the opposite vertex A; then the circle drawn on Ax as diameter is cut orthogonally by the self-conjugate circle. I may also remark that the polars of the feet of the perpendiculars of the triangle, with respect to the above circle, coincide with the lines drawn through the vertices A, B, C parallel to the opposite sides; whence we can easily construct other series of circles cut orthogonally by the self-conjugate circle.

1915. (Proposed by W. H. LAVERTY.)—

$$\left.\begin{aligned}\text{If}\quad C_r &= \frac{1}{x}\cos\frac{r\pi}{2m}+\frac{1}{x^2}\cos\frac{2r\pi}{2m}+\frac{1}{x^3}\cos\frac{3r\pi}{2m}+\&c.\\ \text{and}\quad S_r &= \frac{1}{x}\sin\frac{r\pi}{2m}+\frac{1}{x^2}\sin\frac{2r\pi}{2m}+\frac{1}{x^3}\sin\frac{3r\pi}{2m}+\&c.\end{aligned}\right\}\text{ to }\infty,\text{ show that}$$

$$(P_1)=(C_1^2+S_1^2)(C_3^2+S_3^2)\dots(C_{2m-1}^2+S_{2m-1}^2)=(x^{2m}+1)^{-1},$$

$$(P_2)=(C_0^2+S_0^2)^{\frac{1}{2}}(C_{2m}^2+S_{2m}^2)^{\frac{1}{2}}(C_2^2+S_2^2)\,..\,(C_{2m-2}^2+S_{2m-2}^2)=(x^{2m}-1)^{-1}$$

Solution by S. W. BROMFIELD; REV. J. L. KITCHIN, M.A.; *the* PROPOSER; *and many others.*

Putting θ for $\frac{r\pi}{2m}$, and i as usual for $\surd(-1)$, we have

$$C_r^2+S_r^2=(C_r+iS_r)(C_r-iS_r)=\left(\frac{e^{i\theta}}{x}+\frac{e^{2i\theta}}{x^2}+\dots\right)\left(\frac{e^{-i\theta}}{x}+\dots\right)$$

$$=\frac{e^{i\theta}}{x-e^{i\theta}}\cdot\frac{e^{-i\theta}}{x-e^{-i\theta}}=\frac{1}{x^2-x(e^{i\theta}+e^{-i\theta})+1}=\frac{1}{x^2-2x\cos\theta+1};$$

$$\therefore\quad (P_1)^{-1}=\left(x^2-2x\cos\frac{\pi}{2m}+1\right)\left(x^2-2x\cos\frac{3\pi}{2m}+1\right)\dots=x^{2m}+1;$$

$$(P_2)^{-1}=(x^2-1)\left(x^2-2x\cos\frac{2\pi}{2m}+1\right)\left(x^2-2x\cos\frac{4\pi}{2m}+1\right)\,..=x^{2m}-1.$$

1894. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Supposing n chords to be drawn at random in a given circle, determine the several probabilities that there shall be 0, 1, 2, 3,..... $\frac{1}{2}n(n-1)$ intersections.

Solution by the PROPOSER.

Conceive the circumference of the circle to comprise 2N points equally distributed, the number 2N being indefinite. Then since a system of

chords formed by uniting one of these points with all the others in succession are distributed, in order, at equal angles round the common point, it is evident that a chord inflected at random must have an equal probability of passing through any one of the remaining $2N-1$ points. Therefore as regards the position of the chords in the circle the probabilities are precisely the same whether the chords are drawn at random or supposed to connect pairs of points arbitrarily taken.

Let F_n denote the total number of possible diagrams or configurations of n lines having their extremities chosen from amongst the $2N$ points. To determine the number F_n suppose a set of F_{n-1} diagrams to be drawn, each of them comprising $n-1$ lines and occupying $2n-2$ points. Then to convert any one of these into a diagram having n lines, the $2N-2n+2$ unoccupied points will admit of the formation of $\frac{1}{2}(2N-2n+2)(2N-2n+1) = (N-n+1)(2N-2n+1)$ independent lines, each of which will alike serve for the completion of such diagram. But after following out all these constructions it is evident that each particular diagram must ultimately be reproduced under the separate condition of each of its n lines appearing as the supplementary line, and must thereby become identically repeated n times. Hence by multiplying by the stated number of disposable lines and dividing by n for the purpose of excluding these repetitions, we find

$$\begin{aligned} F_n &= \frac{N-n+1}{n}(2N-2n+1) \,.\, F_{n-1} \\ F_{n-1} &= \frac{N-n+2}{n-1}(2N-2n+3) \,.\, F_{n-2} \\ &\&c. \qquad\qquad \&c. \\ F_1 &= \frac{N}{1}(2N-1) \end{aligned}$$

$$\therefore\ F_n = \frac{N(N-1)\ldots(N-n+1)}{1\,.\,2\ldots n}\Big\{(2N-1)(2N-3)\ldots(2N-2n+1)\Big\} \quad \ldots(p).$$

When N is made equal to n, this becomes

$$f_n = 1\,.\,3\,.\,5\,\ldots\,2n-1\ldots\ldots(m),$$

which is the total number of configurations of n lines occupying $2n$ points.

Again, by multiplying the value of f_n by the number of combinations $2n$ out of $2N$ points, we otherwise derive

$$F_n = \frac{2N(2N-1)\ldots(2N-2n+1)}{1\,.\,2\ldots 2n}(1\,.\,3\,.\,5\ldots 2n-1)\ldots\ldots(q).$$

This result is equivalent to the former; for

$$\frac{N(N-1)\ldots(N-n+1)}{1\,.\,2\ldots n}\Big\{(2N-1)(2N-3)\ldots(2N-2n+1)\Big\}$$

$$= \frac{\underline{|N}}{\underline{|n}\,\underline{|N-n}}\cdot\frac{1\,.\,3\,.\,5\ldots 2N-1}{1\,.\,3\,.\,5\ldots 2N-2n-1} = \frac{\underline{|2N}}{2^n\,\underline{|n}\,\underline{|2N-2n}}$$

$$= \frac{\underline{|2N}}{\underline{|2n}\,\underline{|2N-2n}}\cdot\frac{\underline{|2n}}{2^n\,\underline{|n}} = \frac{\underline{|2N}}{\underline{|2n}\,\underline{|2N-2n}}\,.\,f_n.$$

It will be perceived that in the determination of the latter formula (q) the total number of configurations of the n chords is arrived at by first taking every combination $2n$ out of the $2N$ points; and in the next place, supposing each of these distinct systems of $2n$ points to be separately connected in pairs

in every possible way. Now, as the points are in every case all posited in the periphery of a closed curve, the exterior of which is everywhere convex, a little consideration will show that, as regards intersections and varieties of configuration, each individual system of $2n$ points will yield precisely the same results. It will hence be sufficient to consider only one set of $2n$ points, and the particular position of these points will evidently be quite immaterial. Therefore, generally, whatever be the number 2N, whether finite or infinite, the probabilities required in the question are identical with those which appertain to conditions which are thus simplified and reduced down to the following:—

Let there be given $2n$ points which if joined in consecutive order would form any convex polygon whatever of as many sides; then supposing pairs of these points to be arbitrarily taken and the same to be severally united in n lines, determine the respective probabilities that amongst these n lines there shall be $0, 1, 2, 3, \ldots \frac{1}{2}n(n-1)$ intersections.

Here the problem is entirely divested of considerations which involve the integral calculus, and now appertains to ordinary algebra alone, since the total number of ways is reduced to a finite and determinate number, being in fact the number of ways in which the $2n$ points can be associated in pairs, and this we have already found is expressed by the factorial $1 \,.\, 3 \,.\, 5 \ldots 2n-1$.

When the number n is small, the various combinations may be readily put down, either geometrically or symbolically, and the numbers of intersections respectively counted. To abbreviate and simplify this last operation, let the several points, taken in consecutive order, be denoted by the numerals $1, 2, 3, \ldots 2n$; and in denoting a line let the smaller numeral be always placed first, and let the set of lines which compose each combination or diagram be so arranged that these leading numerals shall proceed in the order of magnitude. When a combination of lines is put down in this convenient manner, the fact of the intersection or non-intersection of any stated line with each of those that follow it will be made apparent by simply noting whether the magnitude of the terminal numeral of the same be included or not between the two numerals which indicate each subsequent line.

As a first example, take the simplest case, viz., that of two chords, which was originally proposed by me as Quest. 1904, in the *Lady's and Gentleman's Diary* for 1856, and solved by Dr. Rutherford and Mr. Miller in the *Diary* for 1857 (pp. 55, 56). According to what precedes, we shall have $2n = 4$ points, and $1 \,.\, 3 = 3$ combinations or diagrams, which may be stated thus:—

12, 34	having	0	intersection
13, 24	,,	1	,,
14, 23	,,	0	,,

As these exhibit two diagrams without intersection and one diagram with intersection, the probabilities of intersection and non-intersection of two arbitrary chords are $\frac{1}{3}$ and $\frac{2}{3}$.

As another example, take the case of three chords, Quest. 1152 of the *Educational Times*, to which an effective solution by the integral calculus is given by Mr. Miller, the Editor, in the *Reprint*, Vol. II., p. 92. Here we have $2n = 6$ points, and $1 \,.\, 3 \,.\, 5 = 15$ combinations, viz.,

12, 34, 56	having	0	intersections
35, 46	,,	1	,,
36, 45	,,	0	,,
13, 24, 56	,,	1	,,
25, 46	,,	2	,,
26, 45	,,	1	,,

14, 23, 56	having	0	intersections
25, 36	,,	3	,,
26, 35	,,	2	,,
15, 23, 46	,,	1	,,
24, 36	,,	2	,,
26, 34	,,	1	,,
16, 23, 45	,,	0	,,
24, 35	,,	1	,,
25, 34	,,	0	,,

This complete system of combinations consists of the following summary

5	diagrams having	0	intersections
6	,,	1	,,
3	,,	2	,,
1	,,	3	,,

Therefore in this case the respective probabilities of 0, 1, 2, 3 intersections are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{1}{15}$.

It will be convenient to express the numbers by means of the coefficients of the several terms of an algebraic function, in which the exponent of x is made to denote the number of intersecting pairs.

The solutions to the two foregoing examples when thus stated are,—

$$I_2 = 2 + x$$
$$I_3 = 5 + 6x + 3x^2 + x^3,$$

in which the coefficient of x^r represents the number of diagrams which have r intersections.

The general solution of the proposed problem will hence be included in a method of determining the function I_n in the general case.

By a somewhat elaborate process I have arrived at the following remarkable results.

Let $1, \delta_1, \delta_2, \delta_3, \ldots \delta_n$ be the first differences of the coefficients of the binomial $(1+x)^{2n}$ taken as far as the central or maximum coefficient; also let

$$\nu = \frac{(n+1)\,n}{2}, \quad \nu' = \frac{n\,(n-1)}{2}, \quad \nu'' = \frac{(n-1)\,(n-2)}{2}, \text{ \&c.}$$

$$\left.\begin{aligned} \text{Then } I_n &= \frac{\delta_n - \delta_{n-1}\,x + \delta_{n-2}\,x^3 - \delta_{n-3}\,x^6 + \delta_{n-4}\,x^{10} - \text{\&c.}}{(1-x)^n} \\ &= \frac{x^{\nu} - \delta_1 x^{\nu'} + \delta_2 x^{\nu''} - \delta_3 x^{\nu'''} + \text{\&c.}}{(x-1)^n} \end{aligned}\right\} \ldots\ldots (a)$$

the result of each division being always an exact function, or one which leaves no remainder. Perhaps the most expeditious form for numerical calculation is

$$I_n = \left(1 - \frac{1}{x}\right)^{-n} \left(x^{\nu-n} - \delta_1 x^{\nu'-n} + \delta_2 x^{\nu''-n} - \delta_3 x^{\nu'''-n} + \text{\&c.}\right)$$
$$= \left(1 + nx^{-1} + n\,\frac{n+1}{2}\,x^{-2} + n\,\frac{n+1}{2}\,\frac{n+2}{3}\,x^{-3} + \text{\&c.}\right)$$
$$\times \left(x^{\nu-n} - \delta_1 x^{\nu'-n} + \delta_2 x^{\nu''-n} - \delta_3 x^{\nu'''-n} + \text{\&c.}\right) \ldots\ldots\ldots\ldots (b)$$

where the coefficients of the first factor, being the ordinary figurate numbers, are easily formed into a preliminary table by successive addition. Also the terms of the second factor may be rejected as soon as the exponent becomes

negative, since the coefficients of negative exponents must necessarily become neutralized in the final result, which will be of the form

$$\mathrm{I}_n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots + c_sx^s,$$

where $s = \dfrac{n(n-1)}{2}$.

The several coefficients of $(1-x)^n \cdot \mathrm{I}_n$ must obviously be the expanded nth differences of the coefficients of I_n, and therefore by (a) these latter must consist of the series of values

$$\delta_n \big| - \delta_{n-1} \big| 0 \big| + \delta_{n-2} \big| 0, 0 \big| - \delta_{n-3} \big| 0, 0, 0 \big| + \delta_{n-4}, \text{\&c.} \ \ldots\ldots (\alpha).$$

The coefficients of I_n may therefore be numerically determined from this series of differences by making n successive summations, which are, in fact, equivalent to n successive divisions by $1-x$. Or the first step of this process may be dispensed with as follows:—Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the first differences of the first n coefficients of $(1+x)^{2n-1}$; then will the $(n-1)$th order of differences of the coefficients of I_n be

$$\alpha_n \big| - \alpha_{n-1} - \alpha_{n-1} \big| + \alpha_{n-2} + \alpha_{n-2} + \alpha_{n-2} \big| - \text{\&c. \&c.} \ \ldots\ldots\ldots (\beta),$$

in which series the signs alternate with each group and the repetitions are successively increased by unity. From these differences the coefficients of I_n are hence determined by $n-1$ successive summations, and the whole operation will occupy just n columns.

As an example of the formula (a) let $n = 5$; then

$$(1+x)^{10} = 1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + \text{\&c.},$$

$$\therefore \text{ values of } \delta_{0\ldots n} = 1, 9, 35, 75, 90, 42;$$

$$\text{and } \mathrm{I}_5 = \frac{42 - 90x + 75x^3 - 35x^6 + 9x^{10} - x^{15}}{(1-x)^5}$$

$$= 42 + 120x + 180x^2 + 195x^3 + 165x^4 + 117x^5 + 70x^6 + 35x^7 + 15x^8 + 5x^9 + x^{10}.$$

Or, by the formula (b)

$$\mathrm{I}_5 = \left(1 + \frac{5}{x} + \frac{15}{x^2} + \frac{35}{x^3} + \frac{70}{x^4} + \frac{126}{x^5} + \frac{210}{x^6} + \frac{330}{x^7} + \frac{495}{x^8} + \frac{715}{x^9} + \frac{1001}{x^{10}} + \text{\&c.}\right) \times (x^{10} - 9x^5 + 35x - \text{\&c.})$$

$$\begin{array}{r}
= x^{10} + 5x^9 + 15x^8 + 35x^7 + 70x^6 + 126x^5 + 210x^4 + 330x^3 + 495x^2 + 715x + 1001 \\
- 9x^5 - 45x^4 - 135x^3 - 315x^2 - 630x - 1134 \\
+ 35x + 175 \\
\hline
= x^{10} + 5x^9 + 15x^8 + 35x^7 + 70x^6 + 117x^5 + 165x^4 + 195x^3 + 180x^2 + 120x + 42
\end{array}$$

And this is undoubtedly the most concise method of obtaining the function I_n when the series of figurate numbers which appear in the first factor are previously tabulated.

To apply the formula (β), we have

$$(1+x)^9 = 1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + \text{\&c.};$$

$$\text{therefore values of } \alpha_{1\ldots n} = 1, 8, 27, 48, 42.$$

Hence by placing the series of differences in vertical columns the successive summations will appear as follows, and the accuracy of the results will be made apparent by their regular subsidence at the end of each column, the same being, in each case, indicated by an asterisk (*) :

α				Diagrams c.	Intersections
42	42	42	42	42	0
-48	-6	36	78	120	1
-48	54	-18	60	180	2
$+27$	-27	45	15	195	3
27	0	45	-30	165	4
$+27$	$+27$	-18	48	117	5
-8	19	$+1$	47	70	6
8	11	12	35	35	7
8	$+3$	15	20	15	8
-8	-5	10	10	5	9
$+1$	-4	$+6$	-4	1	10
1	3	3	-1	*	
1	2	$+1$	*		
1	-1	*			
$+1$	*				

The numbers in each column are successively deduced by the algebraic addition of the corresponding number in the preceding column; and those which extend below the horizontal line may evidently be omitted, in which case the general accuracy of the work will be satisfactorily tested by observing that the last horizontal line should then comprise the coefficients of $(1+x)^{n-1}$ with alternate signs.

If the function I_n be put down with its terms in a reverse order, or as immediately derived from the second of the formulæ (a), thus:

$$I_n = c_s x^s + c_{s-1}\, x^{s-1} + c_{s-2}\, x^{s-2} + \&c.;$$

then the coefficients c_s, c_{s-1}, c_{s-2}, &c., being in a reverse order, their $(n-1)$th order of differences will be the reverse of the foregoing, viz.,

$+1$ (n times); $-\alpha_1$ ($n-1$ times); $+\alpha_2$ ($n-2$ times); &c. &c.

and hence the coefficients may be determined as before.

Thus, returning to the same example:—

α				Diagrams c.	Intersections
$+1$	$+1$	$+1$	$+1$	1	10
1	2	3	4	5	9
1	3	6	10	15	8
1	4	10	20	35	7
$+1$	$+5$	15	35	70	6
-8	-3	12	47	117	5
8	11	$+1$	48	165	4
8	19	-18	$+30$	195	3
-8	-27	45	-15	180	2
$+27$	0	45	60	120	1
27	$+27$	-18	78	42	0
$+27$	54	$+36$	-42	*	
-48	$+6$	42	*		
-48	-42	*			
$+42$	*				

In all these summations it is curious to observe the persistence of negative signs until the final column is attained.

To adduce other examples, and to give a facility of practical reference, the results up to as many as eight chords are shown in the following table:—

TABULATED NUMBER OF CONFIGURATIONS (c).

Number of Chords (n)							Inter-sections
2	3	4	5	6	7	8	
2	5	14	42	132	429	1430	0
1	6	28	120	495	2002	8008	1
	3	28	180	990	5005	24024	2
	1	20	195	1430	9009	51688	3
		10	165	1650	13013	89180	4
		4	117	1617	16016	131040	5
		1	70	1386	17381	169988	6
			35	1056	16991	199264	7
			15	726	15197	214578	8
			5	451	12558	214760	9
			1	252	9646	201460	10
				126	6916	178248	11
				56	4641	149464	12
				21	2912	119168	13
				6	1703	90540	14
				1	924	65640	15
					462	45438	16
					210	30024	17
					84	18908	18
					28	11320	19
					7	6420	20
					1	3432	21
						1716	22
						792	23
						330	24
						120	25
						36	26
						8	27
						1	28
3	15	105	945	10395	135135	2027025	Totals

To find the probability of any proposed number of intersections, the number taken from this table is to be divided by the total given at the foot of the column.

The mathematical relations amongst the coefficients (c) may, according to what has preceded, be expressed under the form of an nth difference. Let r

be any number from 0 to $n\,\dfrac{n-1}{2}$; and

$$c_r - nc_{r-1} + n\,\frac{n-1}{2}\,c_{r-2} - n\,\frac{n-1}{2}\,\frac{n-2}{3}\,c_{r-3} + \&c. = \mathrm{R}.$$

Then when r is of the form $k\,\dfrac{k+1}{2}$, k being any integral number $< n$, we shall have

$$\mathrm{R} = (-1)^k\,\delta_{n-k} = (-)^k\left(\frac{2n\ldots n+k+1}{1\,.\,.\,n-k} - \frac{2n\ldots n+k+2}{1\,.\,.\,n-k-1}\right)$$

$$= (-)^k\,\frac{2k+1}{n+k+1}\cdot\frac{\underline{|2n}}{\underline{|n-k}\;\underline{|n+k}}.$$

And when r is not of the form $k\,\dfrac{k+1}{2}$; then $\mathrm{R} = 0$.

Also $c_r = 0$ for all values of $r < 0$ or $> n\,\dfrac{n-1}{2}$.

Take $r = 0$; it is of the form $k\,\dfrac{k+1}{2}$, and $k = 0$,

therefore
$$c_0 = \mathrm{R} = \frac{\underline{|2n}}{\underline{|n}\;\underline{|n+1}},$$

and hence the probability of the general case of non-intersection is

$$p_0 = \frac{c_0}{1\,.\,3\,.\,5\,.\,.\,2n-1} = \frac{2^n}{\underline{|n+1}}.$$

When $r = 1$ it is again of the form $k\,\dfrac{k+1}{2}$, and $k = 1$;

therefore
$$\mathrm{R} = -\frac{3}{n+2}\cdot\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}},\ \text{and}\ c_1 - nc_0 = \mathrm{R};$$

therefore
$$c_1 = nc_0 + \mathrm{R} = \left(1 - \frac{3}{n+2}\right)\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}$$

$$= \frac{n-1}{n+2}\,\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}} = \frac{\underline{|2n}}{\underline{|n-2}\;\underline{|n+2}};$$

which result exhibits the remarkable property, that the number of configurations having each of them but one intersection is equal to the number of combinations of $n-2$ or $n+2$ out of the $2n$ points.

When $r = 2$ it is not of the form $k\,\dfrac{k+1}{2}$;

therefore
$$\mathrm{R} = 0,\ \text{and}\ c_2 = nc_1 + n\,\frac{n-1}{2}\,c_0 = 0,$$

giving
$$c_2 = n\left(c_1 - \frac{n-1}{2}\,c_0\right) = (n-1)\left(\frac{n}{n+2} - \frac{1}{2}\right)\frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}$$

$$= \frac{n-1)(n-2)}{2(n+2)} \cdot \frac{\lfloor 2n}{\lfloor n-1 \lfloor n+1} = \frac{n-2}{2} \cdot \frac{\lfloor 2n}{\lfloor n-2 \lfloor n+2}.$$

The general combinations for specific intersections may be otherwise deduced more directly from the formula (b). Thus, the coefficients of the binomial $(1+x)^{2n}$ are

$$\beta_n = \frac{n+1\ldots 2n}{1\ldots n},\quad \beta_{n-1} = \frac{n+2\ldots 2n}{1\ldots n-1},\quad \beta_{n-2} = \frac{n+3\ldots 2n}{1\ldots n-2},$$

$$\beta_{n-3} = \frac{n+4\ldots 2n}{1\ldots n-3},\quad \beta_{n-4} = \frac{n+5\ldots 2n}{1\ldots n-4},\ \&c.$$

therefore $\delta_n = \beta_n \quad - \beta_{n-1} = \dfrac{n+2\ldots 2n}{1\ldots n}$

$$\delta_{n-1} = \beta_{n-1} - \beta_{n-2} = \frac{3(n+3\ldots 2n)}{1\ldots n-1} = \frac{3n}{n+2}\,\delta_n$$

$$\delta_{n-2} = \beta_{n-2} - \beta_{n-3} = \frac{5(n+4\ldots 2n)}{1\ldots n-2} = \frac{5n(n-1)}{(n+2)(n+3)}\,\delta_n$$

$$\delta_{n-3} = \beta_{n-3} - \beta_{n-4} = \frac{7(n+5\ldots 2n)}{1\ldots n-3} = \frac{7n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\,\delta_n$$

&c. &c. &c.

Hence, according to (b), we find

$$c_0 = \delta_n = \frac{n+2\ldots 2n}{1\ldots n}$$

$$c_1 = n\delta_n - \delta_{n-1} = \beta_{n-2} = \frac{n+3\ldots 2n}{1\ldots n-2}$$

$$c_2 = n\frac{n+1}{2}\delta_n - n\delta_{n-1} = \frac{n+3\ldots 2n}{2(1\ldots n-3)}$$

$$c_3 = n\frac{n+1}{2}\cdot\frac{n+2}{3}\delta_n - n\frac{n+1}{2}\delta_{n-1} + \delta_{n-2}$$

$$= \frac{n^2+2n-9}{2\,.\,3}\cdot\frac{n+4\ldots 2n}{1\ldots n-3}$$

$$c_4 = \frac{n\ldots n+3}{2\,.\,3\,.\,4}\delta_n - \frac{n(n+1)(n+2)}{2\,.\,3\,.\,4}\delta_{n-1} + n\delta_{n-2}$$

$$= \frac{(n-1)(n+6)}{2\,.\,3\,.\,4}\cdot\frac{n+4\ldots 2n}{1\ldots n-4}$$

$$c_5 = \frac{n\ldots n+4}{2\ldots 5}\delta_n - \frac{n\ldots n+3}{2\,.\,3\,.\,4}\delta_{n-1} + n\frac{n+1}{2}\delta_{n-2}$$

$$= \frac{(n+1)(n-3)(n+8)}{2\,.\,3\,.\,4\,.\,5}\cdot\frac{n+4\ldots 2n}{1\ldots n-4}$$

$$c_6 = \frac{n\ldots n+5}{2\ldots 6}\delta_n - \frac{n\ldots n+4}{2\ldots 5}\delta_{n-1} + \frac{n(n+1)(n+2)}{2\,.\,3}\delta_{n-2} - \delta_{n-3}$$

$$= (n-2)\frac{n^4+14n^3+29n^2-164n-600}{2\,.\,3\,.\,4\,.\,5\,.\,6}\cdot\frac{n+5\ldots 2n}{1\ldots n-4}$$

$$c_7 = \frac{n\ldots n+6}{2\ldots 7}\delta_n - \frac{n\ldots n+5}{2\ldots 6}\delta_{n-1} + \frac{n\ldots n+3}{2.3.4}\delta_{n-2} - n\delta_{n-3}$$

$$= \frac{n^5+19n^4+83n^3-97n^2-1206n+720}{2.3.4.5.6.7} \cdot \frac{n+5\ldots 2n}{1\ldots n-5}$$

&c. &c.

$$c_r = \frac{n\ldots n+r-1}{1\ldots r}\delta_n - \frac{n\ldots n+r-2}{1\ldots r-1}\delta_{n-1} + \frac{n\ldots n+r-4}{1\ldots r-3}\delta_{n-2}$$

$$- \frac{n\ldots n+r-7}{1\ldots r-6}\delta_{n-3} + \&\text{c.}$$

where the coefficients of δ_n, δ_{n-1}, δ_{n-2}, &c., are figurate numbers which first descend one degree, then two degrees, then three degrees, and so on, by a uniformly widening interval of succession.

By dividing these values of c by

$$\Sigma c = 1.3.5\ldots 2n-1 = \frac{1\ldots 2n}{2^n(1\ldots n)} = \frac{n+1\ldots 2n}{2^n}$$

the corresponding probabilities, for stated intersections, are finally determined to be

$$p_0 = \frac{2^n}{\underline{|n+1}}$$

$$p_1 = \frac{1}{(n+1)(n+2)} \cdot \frac{2^n}{\underline{|n-2}}$$

$$p_2 = \frac{1}{(n+1)(n+2)} \cdot \frac{2^{n-1}}{\underline{|n-3}}$$

$$p_3 = \frac{n^2+2n-9}{(n+1)(n+2)(n+3)} \cdot \frac{2^{n-1}}{3\underline{|n-3}}$$

$$p_4 = \frac{(n-1)(n+6)}{(n+1)(n+2)(n+3)} \cdot \frac{2^{n-3}}{3\underline{|n-4}}$$

$$p_5 = \frac{(n-3)(n+8)}{(n+2)(n+3)} \cdot \frac{2^{n-3}}{15\underline{|n-4}}$$

$$p_6 = \frac{n^4+14n^3+29n^2-164n-600}{(n+1)(n+2)(n+3)(n+4)} \cdot \frac{2^{n-4}(n-2)}{45\underline{|n-4}}$$

$$p_7 = \frac{n^5+19n^4+83n^3-97n^2-1206n+720}{(n+1)(n+2)(n+3)(n+4)} \cdot \frac{2^{n-4}}{315\underline{|n-5}}$$

&c. &c.

$$p_r = \frac{(r+1\ldots r+n-1)\delta_n - (r\ldots r+n-2)\delta_{n-1} + (r-2\ldots r+n-4)\delta_{n-2} - \&\text{c.}}{(1\ldots n-1)(1.3\ldots 2n-1)}$$

The general case of non-intersection may be otherwise calculated, for successive values of n, by another method in which the reasoning is of the most palpable and elementary kind. Suppose a set of diagrams to be formed from $2(n+1)$ points, each of them comprising $n+1$ non-intersecting lines, and let the number of these diagrams be denoted by ϕ_{n+1}. All the varieties of diagram, which make up the set, may obviously be attained, inclusively,

by the following considerations. First, conceive a certain partition of diagrams to be chosen such that all of them shall contain one particular line, and refer these several diagrams to this common line, which may be designated the $(n+1)$th line. Then, as there is no intersection, it will appear that the other n lines must be either wholly on one side, or distributed in a specific manner on both sides of this $(n+1)$th line. That is, if there be x lines on one side of it, there will be $n-x$ lines on the other side, and the number x, to include all partitions, may have the several values $0, 1, 2, 3, \ldots n$. Now, following out our notation, the x non-intersecting lines, which unite $2x$ points on one side, may be drawn in ϕ_x ways; and the $n-x$ lines on the other side may in like manner be drawn in ϕ_{n-x} ways. And as each of the former of these constructions may be united with each of the latter, the number of complete diagrams which thereby result, under the stated hypothesis of distribution. will be $\phi_x\phi_{n-x}$. Hence by giving to x all its possible values, as before stated, the total number of non-intersecting diagrams is found to be

$$\phi_{n+1} = \Sigma\phi_x\phi_{n-x}$$
$$= \phi_0\phi_n + \phi_1\phi_{n-1} + \phi_2\phi_{n-2} + \ldots + \phi_n\phi_0$$

in which $\phi_0 = 1$.

By means of this general formula the value of the function for any proposed number is immediately deduced from the values previously determined for all inferior numbers. Thus we get, successively,

$$\begin{aligned}
\phi_1 &= 1.1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 1 \\
\phi_2 &= 1.1+1.1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 2 \\
\phi_3 &= 1.2+1.1+2.1 \ldots\ldots\ldots\ldots\ldots\ldots = 5 \\
\phi_4 &= 1.5+1.2+2.1+5.1 \ldots\ldots\ldots\ldots = 14 \\
\phi_5 &= 1.14+1.5+2.2+5.1+14.1 \ldots\ldots = 42 \\
\phi_6 &= 1.42+1.14+2.5+5.2+14.1+42.1 = 132
\end{aligned}$$

&c. &c. &c.

To determine f_{n+1} the total number of diagrams, both intersecting and non-intersecting, we observe that, as the several lines may or may not intersect the $(n+1)$th line, it is no longer requisite to have a distinct set of lines, or an even number of points, on each side of it. The points may therefore be distributed in any manner on each side of the $(n+1)$th line. If there be z points on one side of it, there will be $2n-z$ points on the other side, and the number z may take any of the values $0, 1, 2, 3, \ldots 2n$. Also under each hypothesis of distribution the z and $2n-z$ points respectively on each side of the $(n+1)$th line make up a system of $2n$ points which may be joined in pairs by n lines in f_n ways, the same being independent of the value of z. Therefore, as z admits of having $2n+1$ values, the total number of diagrams of $n+1$ lines, occupying $2(n+1)$ points, under every arrangement, is

$$f_{n+1} = (2n+1)\,f_n$$

from which we readily infer that

$$f_{n+1} = 1.3.5 \ldots\ldots (2n+1)$$
$$f_n \quad = 1.3.5 \ldots\ldots (2n-1)$$

the same result as before obtained.

The probability of non-intersection is of course found by dividing the value of ϕ by the corresponding value of f.

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